

VISCOUS SINGULAR SHOCK PROFILES FOR A SYSTEM OF CONSERVATION LAWS MODELING TWO-PHASE FLOW

TING-HAO HSU

December 2, 2015

ABSTRACT. This paper is concerned with singular shocks for a system of conservation laws modeling incompressible two-phase fluid flow. We prove the existence of viscous profiles using the geometric singular perturbation theory. Weak convergence and growth rates of the unbounded family of solutions are also obtained.

1. INTRODUCTION

Keyfitz et al [KSS03, KSZ04] considered the system of conservation laws

$$(1.1) \quad \begin{aligned} \beta_t + (vB_1(\beta))_x &= 0 \\ v_t + (v^2B_2(\beta))_x &= 0 \end{aligned}$$

where $t \geq 0$, $x \in \mathbb{R}$, $v \in \mathbb{R}$, $\beta \in [\rho_1, \rho_2]$ with $\rho_2 < \rho_1$ and

$$(1.2) \quad B_1(\beta) = \frac{(\beta - \rho_1)(\beta - \rho_2)}{\beta}, \quad B_2(\beta) = \frac{\beta^2 - \rho_1\rho_2}{2\beta^2}.$$

For Riemann problems with data in feasible regions, they constructed uniquely defined admissible solutions. It can be readily shown that this system is not everywhere hyperbolic, and hence standard methods does not apply (see e.g. [Smo83, Daf10]). To resolve this problem, along with rarefaction waves and regular shocks, the concept of singular shocks was adopted. A singular shock solution, roughly speaking, is a distribution which contains delta measures and is the weak limit of a sequence of approximate viscous solutions. For details of the definition, we refer to [Sev07, Key11].

The existence of singular shocks for (1.1) was proved in [KSZ04]. In that work, for certain Riemann data

$$(1.3) \quad (\beta, v)(x, 0) = \begin{cases} (\beta_L, v_L), & x < 0 \\ (\beta_R, v_R), & x > 0 \end{cases}$$

approximate solutions of the regularized system via Dafermos regularization

$$(1.4\epsilon) \quad \begin{aligned} \beta_t + (vB_1(\beta))_x &= \epsilon t \beta_{xx} \\ v_t + (v^2B_2(\beta))_x &= \epsilon t v_{xx} \end{aligned}$$

were constructed. A family of exact solutions of (1.4 ϵ) and (1.3), rather than approximate solutions, is called a *viscous profile*. In this paper, we prove existence of viscous profile, also we give descriptions of their limiting behavior including weak convergence and growth rates. The main tool in our study is the *Geometric Singular Perturbation Theory* (GSPT), which will be introduced in later sections. The use of this tool on singular shocks was first introduced in the pioneering work of Schecter [Sch04].

The system (1.1) is equivalent to a two-fluid model for incompressible two-phase flow [DP99, p.248] of the form

$$(1.5) \quad \begin{aligned} \partial_t(\alpha_i) + \partial_x(\alpha_i u_i) &= 0 \\ \partial_t(\alpha_i \rho_i u_i) + \partial_x(\alpha_i \rho_i u_i^2) + \alpha_i \partial_x p_i &= F_i, \quad i = 1, 2, \end{aligned}$$

2010 *Mathematics Subject Classification.* 35L65, 35L67, 34E15, 34C37.

Key words and phrases. Conservation laws; Singular shocks; Viscous profiles; Dafermos regularization; Geometric Singular Perturbation Theory.

where the drag terms F_i are neglected and the pressure terms satisfy $p_1 = p_2$. To reduce (1.5) to (1.1), in [KSS03] the volume fractions α_1 and $\alpha_2 = 1 - \alpha_1$ have been replaced by a density-weighted volume element $\beta = \rho_2\alpha_1 + \rho_1\alpha_2$ and the momentum equations replaced by a single equation for the momentum difference $v = \rho_1u_1 - \rho_2u_2 - (\rho_1 - \rho_2)K$, where $K = \alpha_2u_1 + \alpha_1u_2$ is taken to be zero. This is a simple example of continuous model for two-phase flow, but it shares with other continuous models the property of changing type – that is, it is not hyperbolic for some (in this case, most) states.

The purpose of this study is to shed light on the mathematical properties of the change-of-type system that appear in continuous models of two-phase flow. The original studies [KSS03, KSZ04] showed the existence of self-similar solutions with reasonable properties. Specifically, the singular shocks that appear can be considered to be propagating phase boundaries. In this paper, we focus on viscous profiles of singular shocks and unveil some of their limiting behavior.

In Section 2, we state our main result, and in Section 3 the validity of the assumptions of the theorem is discussed, with some proofs for the sufficient conditions postponed to Section 8. In Section 4, we recall and enhance some tools in GSPT, including Fenichel's Theorems and the Exchange Lemma. Section 5 is devoted to describing the structure of the system. The proof of the main theorem is completed in Section 6, and numerical simulations are shown in Section 7.

2. MAIN RESULT

In standard notation for conservation laws, we write (1.1) as

$$(2.6) \quad u_t + f(u)_x = 0,$$

where $u = (\beta, v)$, and write Riemann data for Riemann problems in the form

$$(2.7) \quad u(x, 0) = u_L + (u_R - u_L)H(x),$$

where $H(x)$ is the step function taking value 0 if $x < 0$; 1 if $x > 0$.

We study the systems that approximate (2.6) via the Dafermos regularization:

$$(2.8\epsilon) \quad u_t + f(u)_x = \epsilon u_{xx}$$

for small $\epsilon > 0$. Using the self-similar variable $\xi = x/t$, the system is converted to

$$(2.9\epsilon) \quad -\xi \frac{d}{d\xi} u + \frac{d}{d\xi} (f(u)) = \epsilon \frac{d^2}{d\xi^2} u,$$

and the initial condition (2.7) becomes

$$(2.10) \quad u(-\infty) = u_L, \quad u(+\infty) = u_R.$$

The system (2.9 ϵ) is equivalent to

$$(2.11\epsilon) \quad \begin{aligned} -\epsilon u_\xi &= f(u) - \xi u - w \\ w_\xi &= -u \end{aligned}$$

or, up to a rescaling of time,

$$(2.12\epsilon) \quad \begin{aligned} \dot{u} &= f(u) - \xi u - w \\ \dot{w} &= -\epsilon u \\ \dot{\xi} &= \epsilon. \end{aligned}$$

The time variable in (2.12 ϵ) is implicitly defined by the equation of $\dot{\xi}$. When $\epsilon = 0$, (2.12 ϵ) is reduced to

$$(2.13) \quad \begin{aligned} \dot{u} &= f(u) - \xi u - w \\ \dot{w} &= 0, \quad \dot{\xi} = 0. \end{aligned}$$

Returning to the (β, v) notation, the system (2.12 ϵ) is written as

$$(2.14\epsilon) \quad \begin{aligned} \dot{\beta} &= -B_1(\beta)v - \xi\beta - w_1 \\ \dot{v} &= -B_2(\beta)v^2 - \xi v - w_2 \\ \dot{w}_1 &= -\epsilon\beta \\ \dot{w}_2 &= -\epsilon v \\ \dot{\xi} &= \epsilon, \end{aligned}$$

and (2.13) becomes

$$(2.15) \quad \begin{aligned} \dot{\beta} &= -B_1(\beta)v - \xi\beta - w_1 \\ \dot{v} &= -B_2(\beta)v^2 - \xi v - w_2 \\ \dot{w}_1 &= 0, \quad \dot{w}_2 = 0, \quad \dot{\xi} = 0. \end{aligned}$$

The linearization at any equilibrium $(\beta, v, w_1, w_2, \xi)$ for (2.15) has eigenvalues $\lambda_{\pm}(\beta, v) - \xi$, where

$$(2.16) \quad \lambda_{\pm}(u) = 2vB_2(\beta) \pm v\sqrt{B_1(\beta)B_2'(\beta)}.$$

Note that $\text{Re}(\lambda_{\pm}(u)) = 2vB_2(\beta)$ since $B_1(\beta)B_2'(\beta) \leq 0$ when $\rho_2 \leq \beta \leq \rho_1$. Moreover, the system is nonhyperbolic everywhere in the physical region except on the union of the lines $\{\beta = \rho_1\}$, $\{\beta = \rho_2\}$, and $\{v = 0\}$.

An *over-compressive shock region* is a region where the condition (H1) defined below holds. It was shown in [KSZ04] that any data in an over-compressive shock region admits a singular shock solution, and the shock speed s is defined by (2.17) below. Our main theorem confirms Dafermos profiles in a subset of this region.

Main Theorem. *Consider the Riemann problem (1.1), (1.3). Let $u_L = (\beta_L, v_L)$ and $u_R = (\beta_R, v_R)$ be two points in $[\rho_1, \rho_2] \times (0, \infty)$ with $\beta_R \neq \beta_L$, and let*

$$(2.17) \quad s = \frac{v_L B_1(\beta_L) - v_R B_1(\beta_R)}{\beta_L - \beta_R}$$

$$(2.18) \quad w_L = f(u_L) - s u_L, \quad w_R = f(u_R) - s u_R$$

$$(2.19) \quad e_0 = w_{2L} - w_{2R}$$

where we denote $w_L = (w_{1L}, w_{2L})$ and $w_R = (w_{1R}, w_{2R})$. Assume

(H1) $\text{Re}(\lambda_{\pm}(u_R)) < s < \text{Re}(\lambda_{\pm}(u_L))$, where $\lambda_{\pm}(u)$ are defined in (2.16).

(H2) $e_0 > 0$.

(H3) For the system (2.15), there exists a trajectory joining (β_L, v_L, w_L, s) and $(\rho_1, +\infty, w_L, s)$, and a trajectory joining (β_R, v_R, w_R, s) and $(\rho_2, +\infty, w_R, s)$.

Then there is a singular shock with Dafermos profile for the Riemann data (u_L, u_R) . That is, for each small $\epsilon > 0$, there is a solution $\tilde{u}^\epsilon(\xi)$ of (2.9 ϵ) and (2.10), and $\tilde{u}_\epsilon(\xi)$ becomes unbounded as $\epsilon \rightarrow 0$. Indeed,

$$(2.20) \quad \max_{\xi} \left(\epsilon \log \tilde{v}_\epsilon(\xi) \right) = \frac{(\rho_1 - \rho_2)(w_{2L} - w_{2R})}{\rho_1 + \rho_2} + o(1) \quad \text{as } \epsilon \rightarrow 0.$$

Moreover, if we set $u_\epsilon(x, t) = \tilde{u}_\epsilon(x/t)$, then $u_\epsilon(x, t)$ is a solution of (2.8 ϵ) and

$$(2.21a) \quad \beta_\epsilon \rightharpoonup \beta_L + (\beta_R - \beta_L)H(x - st)$$

$$(2.21b) \quad v_\epsilon \rightharpoonup v_L + (v_R - v_L)H(x - st) + \frac{e_0}{\sqrt{1 + s^2}} t \delta_{\{x=st\}}$$

in the sense of distributions as $\epsilon \rightarrow 0$.

The trajectories in (H3) are illustrated in Fig 1.

Remark 1. A similar result holds if $v_L < 0$ and $v_R < 0$. In that case, the assumption $e_0 > 0$ in (H2) is replaced by $e_0 < 0$, and $+\infty$ in (H3) is replaced by $-\infty$.

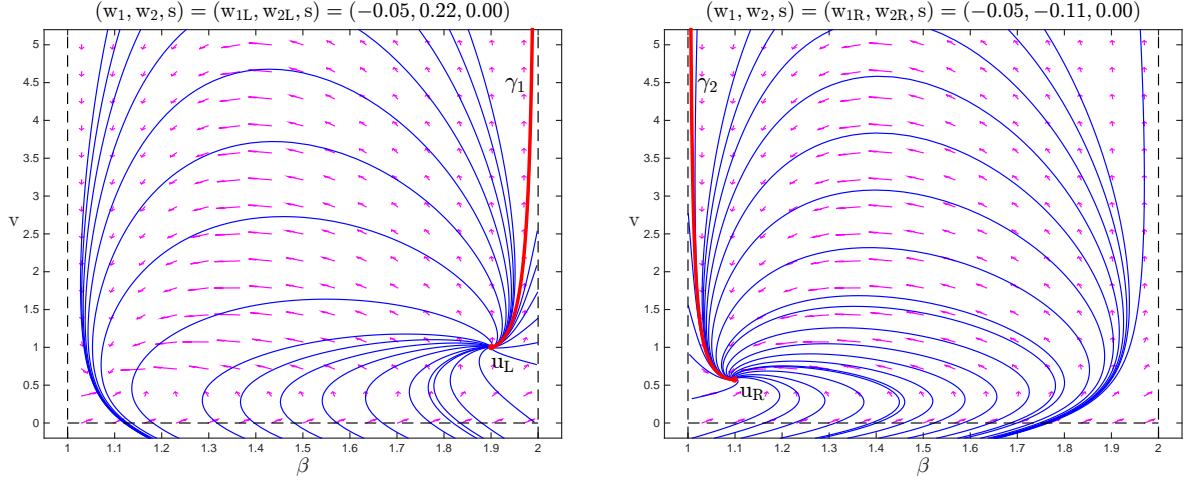


FIGURE 1. Phase portraits for $\dot{u} = f(u) - su - w$ with fixed s and w . The singular trajectories in (H3) are denoted by γ_1 and γ_2 .

The notation $t\delta_{\{x=st\}}$ in (2.21b) denotes, following [TZZ94, CL03], the functional on $C_c^\infty(\mathbb{R} \times \mathbb{R}^+)$ defined by

$$(2.22) \quad \langle t\delta_{\{x=st\}}, \varphi \rangle = \int_0^\infty t\varphi(st, t) \sqrt{1+s^2} dt.$$

The weight $\sqrt{1+s^2}$ in the integral is to normalize the functional so that it is independent of parametrization of the line $\{x=st\}$.

The estimate (2.20) confirms the asymptotic behavior conjectured in [KSS03]. In [KSZ04], some approximate solutions for the Dafermos regularization were constructed, but they were not exact solutions to (2.9 ϵ). The results in the main theorem can also be compared to [Sch04] and [KT12], where Dafermos profiles were constructed for a system motivated by gas dynamics. Those authors obtained families of unbounded solutions to (2.9 ϵ), but they did not give descriptions of asymptotic behaviors of solutions.

The assumption (H3) says that there exist solutions of (2.15) of the form

$$(2.23) \quad \gamma_1 = (\beta_1(\xi), v_1(\xi), w_{1L}, w_{2L}, s), \quad \gamma_2 = (\beta_2(\xi), v_2(\xi), w_{1R}, w_{2R}, s)$$

satisfying

$$(2.24) \quad \lim_{\xi \rightarrow -\infty} (\beta_1(\xi), v_1(\xi)) = (\beta_L, v_L), \quad \lim_{\xi \rightarrow \infty} (\beta_1(\xi), v_1(\xi)) = (\rho_1, +\infty)$$

and

$$(2.25) \quad \lim_{\xi \rightarrow -\infty} (\beta_2(\xi), v_2(\xi)) = (\rho_2, +\infty), \quad \lim_{\xi \rightarrow \infty} (\beta_2(\xi), v_2(\xi)) = (\beta_R, v_R).$$

A local analysis for (2.15) with $(w, \xi) = (w_L, s)$ and (w_R, s) , respectively, at $(\rho_1, +\infty)$ and $(\rho_2, +\infty)$ shows that the trajectories in (H3), if they exist, are unique.

A sample set of data for which (H1)-(H3) holds is, following [KSS03],

$$(2.26) \quad \rho_1 = 2, \quad \rho_2 = 1, \quad u_L = (1.9, 1.0), \quad u_R = (1.1, 1.1/1.9).$$

This will be verified in the next subsection.

3. SUFFICIENT CONDITIONS FOR (H1)-(H3)

The regions at which (H1) holds, or the over-compressive shock regions, can be described by the following

Proposition 3.1. *In the Riemann problem (1.1), (2.7), let $u_L = (\beta_L, v_L)$ and $u_R = (\beta_R, v_R)$ be two points in $[\rho_1, \rho_2] \times (0, \infty)$. Then (H1) holds if and only if u_R lies in the interior of a cusped triangular region*

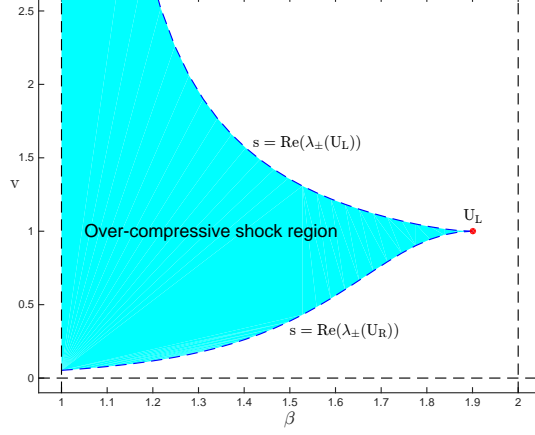


FIGURE 2. The over-compressive shock region for $\rho_1 = 2$, $\rho_2 = 1$, $U_L = (1.1, 1.1/1.9)$.

bounded by the curves

$$(3.27) \quad v = v_L \left(\frac{B_1(\beta_L) - 2B_2(\beta_L)(\beta_L - \beta)}{B_1(\beta)} \right), \quad \rho_2 < \beta_R < \beta_L,$$

and

$$(3.28) \quad v = v_L \left(\frac{B_1(\beta_L)}{B_1(\beta) + 2B_2(\beta)(\beta_L - \beta)} \right), \quad \rho_2 < \beta_R < \beta_L.$$

On the boundary segment (3.27), $s = \text{Re}(\lambda_{\pm}(u_L))$, and on (3.28), $s = \text{Re}(\lambda_{\pm}(u_R))$.

The curves defined by (3.27) and (3.28) and the region where over-compressive shock solution exist are illustrated in Fig 2.

Proof. This follows from a direct calculation. See [KSS03, Corollary 3.1]. \square

The following proposition asserts that (H2) is implied by (H1).

Proposition 3.2. *In the Riemann problem (1.1), (2.7), if the Riemann data lie in an over-compressive shock region in $[\rho_1, \rho_2] \times (0, \infty)$, then (H2) holds.*

Proof. See [KSS03, Section 3.1]. \square

The assumption (H3) is a condition on dynamics of 2-dimensional systems. Analyzing phase portraits we have the following

Proposition 3.3. *Given Riemann data in an over-compressive shock region in $[\rho_1, \rho_2] \times (0, \infty)$, if $\beta_R < \sqrt{\rho_1 \rho_2} < \beta_L$, $w_{1L} < 0$, $w_{2R} < 0 < w_{2L}$, and $|s|$ is sufficiently small, then (H3) holds.*

Proof. See Section 8. \square

Proposition 3.2 says that (H2) holds whenever (H1) holds, so the Main Theorem requires only (H1) and (H3). The author believes that (H3) is also a consequence of (H1). This needs further work to be verified.

For the sample set of data (2.26), we have

$$(3.29) \quad (w_{1L}, w_{2L}) = (-.05, .22), \quad (w_{1R}, w_{2R}) = (-.05, -.11), \quad s = 0.$$

Since $w_{2R} < w_{2L}$, (H2) holds. From Proposition 3.1 and 3.3, (H1) and (H3) also hold. Hence the main theorem applies. Note that the conditions (H1)-(H3) persist under perturbation of the Riemann data (u_L, u_R) , so those assumptions still hold for any data close to (2.26).

4. GEOMETRIC SINGULAR PERTURBATION THEORY

Our main goal is to solve the boundary value problem (2.9 ϵ) and (2.10). Note that (2.9 ϵ) is a *singularly perturbed equation* since the perturbation $\epsilon \frac{d^2}{dt^2} u$ has a higher order derivative than the other terms in the equation. To deal with singularly perturbed equations, we will apply *Geometric Singular Perturbation Theory* (GSPT). The idea of GSPT is to first study a set of subsystems which forms a decomposition of a system, and then to use the information for the subsystems to conclude results for the original system. Prototypical examples include relaxation oscillations for forced Van der Pol Equations [DR96, KS01a, KS01b] and FitzHugh-Nagumo Equations [JKL91, KSS03, LVV06]. Surveys on this topic can be found in [Jon95, Kap99, KJ01, RT02].

In Section 4.1 and 4.2, we recall some fundamental theorems in GSPT. In Section 4.3 we state and give new proofs for a version of the Exchange Lemma.

4.1. Fenichel's Theory for Fast-Slow Systems. Note that (2.12 ϵ) is a *fast-slow system*, which means that the system is of the form

$$(4.30\epsilon) \quad \begin{aligned} \dot{x} &= f(x, y, \epsilon) \\ \dot{y} &= \epsilon g(x, y, \epsilon) \end{aligned}$$

where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$, and ϵ is a parameter. In order to deal with fast-slow systems, Fenichel's Theory was developed in [Fen74, Fen77, Fen79]. Some expositions for that theory can be found in [Wig94, Jon95].

An important feature of a fast-slow system is that the system can be decomposed into two subsystems: the *limiting fast system* and the *limiting slow system*. The limiting fast system is obtained by taking $\epsilon = 0$ in (4.30 ϵ); that is,

$$(4.31) \quad \begin{aligned} \dot{x} &= f(x, y, 0) \\ \dot{y} &= 0. \end{aligned}$$

On the other hand, note that the system (4.30 ϵ) can be converted to, after a rescaling of time,

$$(4.32\epsilon) \quad \begin{aligned} \epsilon x' &= f(x, y, \epsilon) \\ y' &= g(x, y, \epsilon). \end{aligned}$$

Taking $\epsilon = 0$ in (4.32 ϵ), we obtain the limiting slow system

$$(4.33) \quad \begin{aligned} 0 &= f(x, y, 0) \\ y' &= g(x, y, 0). \end{aligned}$$

Note that the limiting slow system (4.33) describes dynamics on the set of critical points of the limiting fast system (4.31), so we will need to piece together the information of the limiting fast system and the limiting slow system in the vicinity of the set of critical points. To piece this information together, *normal hyperbolicity* defined below will be a crucial condition.

Definition 1. A *critical manifold* \mathcal{S}_0 for (4.31) is an l -dimensional manifold consisting of critical points of (4.31). A critical manifold is *normally hyperbolic* if $D_x f(x, y, 0)|_{\mathcal{S}_0}$ is hyperbolic. That is, at any point $(x_0, y_0) \in \mathcal{S}_0$, all eigenvalues of $D_x f(x, y, 0)|_{(x_0, y_0)}$ have nonzero real part.

Now we turn to discussing normal hyperbolicity for general systems

$$(4.34) \quad \dot{z} = F(z),$$

where $z \in \mathbb{R}^N$, $N \geq 1$. A manifold $\mathcal{S} \subset \mathbb{R}^N$ is *locally invariant* if for any point $p \in \mathcal{S} \setminus \partial\mathcal{S}$, there exist $t_1 < 0 < t_2$ such that $p \cdot (t_1, t_2) \in \mathcal{S}$, where \cdot denotes the flow for (4.34). In the vicinity of a locally invariant manifold, under certain conditions the system can be decomposed into lower-dimensional subsystems. For instance, when $\mathcal{S} = \{p_0\}$ is an isolated hyperbolic equilibrium for (4.34), the stable and unstable manifolds $W^s(p_0)$ and $W^u(p_0)$ exist according to the Hartman-Grobman Theorem [Har64], and the union of their tangent spaces at p_0 spans \mathbb{R}^N .

A locally invariant C^r manifold $\mathcal{S} \subset \mathbb{R}^N$, $r \geq 1$, is normally hyperbolic for the system (4.34) if the growth rate of vectors transverse to the manifold dominates the growth rate of vectors tangent to the manifold. (Note that this is consistent with Definition 1.) In this case, from the standard theory for normally hyperbolic

manifolds (see, for example, [HPS77, VvG87, CL88]) it is assured that stable and unstable manifolds $W^s(\mathcal{S})$ and $W^u(\mathcal{S})$ are defined.

For a locally invariant manifold $\Lambda \subset \mathbb{R}^N$ for (4.34) which is not necessarily normally hyperbolic, a *center manifold* is a normally hyperbolic locally invariant manifold, with the smallest possible dimension, containing Λ . In classical cases, $\Lambda = \{p_0\}$ is an isolated non-hyperbolic equilibrium, and a center manifold for p_0 has dimension equal to the number of generalized eigenvalues of $DF(p_0)$ with zero real part. For instance, the planar system

$$\dot{x} = x^3, \quad \dot{y} = y,$$

has a non-hyperbolic isolated equilibrium $p_0 = (0, 0)$, and the x -axis is a center manifold for p_0 . For general invariant sets Λ , we refer to [CLY00a, CLY00b].

Fenichel's Theory is a center manifold theory for fast-slow systems. For a normally hyperbolic critical manifold \mathcal{S}_0 for (4.31), the stable and unstable manifolds $W^s(\mathcal{S}_0)$ and $W^u(\mathcal{S}_0)$ can be defined in the natural way. We denote them by $W_0^s(\mathcal{S}_0)$ and $W_0^u(\mathcal{S}_0)$ to indicate their invariance under (4.30 ϵ) with $\epsilon = 0$. Fenichel's Theory assures that the hyperbolic structure of \mathcal{S}_0 persists under perturbation (4.30 ϵ). Below we state three fundamental theorems of Fenichel's Theory following [Jon95].

Theorem 4.1 (Fenichel's Theorem 1). *Consider the system (4.30 ϵ), where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$, and f, g are C^r for some $r \geq 2$. Let \mathcal{S}_0 be a compact normally hyperbolic manifold for (4.31). Then for any small $\epsilon \geq 0$ there exist locally invariant C^r manifolds, denoted by \mathcal{S}_ϵ , $W_\epsilon^s(\mathcal{S}_\epsilon)$ and $W_\epsilon^u(\mathcal{S}_\epsilon)$, which are C^1 $O(\epsilon)$ -close to \mathcal{S}_0 , $W_0^s(\mathcal{S}_0)$ and $W_0^u(\mathcal{S}_0)$, respectively. Moreover, for any continuous families of compact sets $\mathcal{I}_\epsilon \subset W_\epsilon^u(\mathcal{S}_\epsilon)$, $\mathcal{J}_\epsilon \subset W_\epsilon^s(\mathcal{S}_\epsilon)$, $\epsilon \in [0, \epsilon_0]$, there exist positive constants C and ν such that*

$$(4.35a) \quad \text{dist}(z \cdot t, \mathcal{S}_\epsilon) \leq Ce^{\nu t} \quad \forall z \in \mathcal{I}_\epsilon, t \leq 0$$

$$(4.35b) \quad \text{dist}(z \cdot t, \mathcal{S}_\epsilon) \leq Ce^{-\nu t} \quad \forall z \in \mathcal{J}_\epsilon, t \geq 0,$$

where \cdot denotes the flow for (4.30 ϵ).

Proof. See [Jon95, Theorem 3]. □

Remark 2. If \mathcal{S}_0 is locally invariant under (4.30 ϵ) for each ϵ , then the \mathcal{S}_ϵ can be chosen to be \mathcal{S}_0 because of the construction in the proof of [Jon95, Theorem 3].

Note that $W_\epsilon^u(\mathcal{S}_\epsilon)$ and $W_\epsilon^s(\mathcal{S}_\epsilon)$ can be interpreted as a decomposition in a neighborhood of \mathcal{S}_0 in (x, y) -space. The following theorem asserts that this induces a change of coordinates (a, b, c) such that $W_\epsilon^u(\mathcal{S}_\epsilon)$ and $W_\epsilon^s(\mathcal{S}_\epsilon)$ correspond to (a, c) -space and (b, c) -space, respectively.

Theorem 4.2 (Fenichel's Theorem 2). *Suppose the assumptions in Theorem 4.1 hold. Then under a C^r ϵ -dependent coordinate change $(x, y) \mapsto (a, b, c)$, the system (4.30 ϵ) can be brought to the form*

$$(4.36\epsilon) \quad \begin{aligned} \dot{a} &= A^u(a, b, c, \epsilon)a \\ \dot{b} &= A^s(a, b, c, \epsilon)b \\ \dot{c} &= \epsilon(h(c) + E(a, b, c, \epsilon)) \end{aligned}$$

in a neighborhood of \mathcal{S}_ϵ , where the coefficients are C^{r-2} functions satisfying

$$(4.37) \quad \inf_{\lambda \in \text{Spec} A^u(a, b, c, 0)} \text{Re } \lambda > 2\nu, \quad \sup_{\lambda \in \text{Spec} A^s(a, b, c, 0)} \text{Re } \lambda < -2\nu$$

for some $\nu > 0$ and

$$(4.38) \quad E = 0 \quad \text{on } \{a = 0\} \cup \{b = 0\}.$$

Proof. See [Jon95, Section 3.5] or [JT09, Proposition 1]. □

The family of trajectories for (4.33) forms a foliation of \mathcal{S}_0 . The following theorem says that this induces a foliation of $W_\epsilon^u(\mathcal{S}_\epsilon)$ and $W_\epsilon^s(\mathcal{S}_\epsilon)$.

Theorem 4.3 (Fenichel's Theorem 3). *Suppose the assumptions in Theorem 4.1 hold. Let Λ_0 be a submanifold in \mathcal{S}_0 which is locally invariant under (4.33). Then there exist locally invariant manifolds Λ_ϵ , $W_\epsilon^s(\Lambda_\epsilon)$, and $W_\epsilon^u(\Lambda_\epsilon)$ for (4.30 ϵ) which are C^{r-2} $O(\epsilon)$ -close to Λ_0 , $W_0^s(\Lambda_0)$, and $W_0^u(\Lambda_0)$, respectively. Moreover, for any continuous families of compact sets $\mathcal{I}_\epsilon \subset W_\epsilon^u(\Lambda_\epsilon)$, $\mathcal{J}_\epsilon \subset W_\epsilon^s(\Lambda_\epsilon)$, $\epsilon \in [0, \epsilon_0]$, there exist positive*

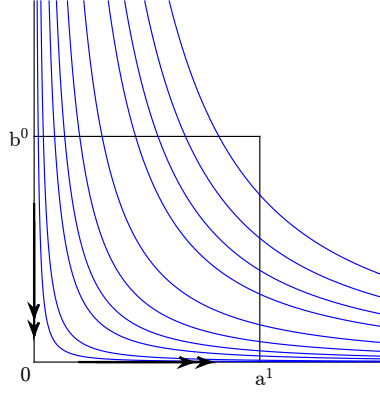


FIGURE 3. Trajectories in the rectangle $\{0 \leq a \leq a^1, 0 \leq b \leq b^0\}$ can be parametrized in $T \geq 0$ by $a(T) = a^1$, $b(0) = b^0$.

constants C and ν such that (4.35) holds with \mathcal{S}_ϵ replaced by Λ_ϵ . Suppose in addition that S_0 is invariant under (4.30 ϵ) for each ϵ . Then Λ_ϵ can be chosen to be Λ_0 .

Proof. Using Fenichel's coordinates (a, b, c) in Theorem 4.2 for the splitting of \mathcal{S}_0 , we can take $W_\epsilon^u(\Lambda_\epsilon)$ and $W_\epsilon^s(\Lambda_\epsilon)$ to be the pre-images of the sets $\{(a, b, c) : a = 0, c \in \Lambda_0\}$ and $\{(a, b, c) : b = 0, c \in \Lambda_0\}$, respectively, in (x, y) -space. From (4.37) we obtain (4.35) with \mathcal{S}_ϵ replaced by Λ_ϵ . Suppose S_0 is invariant under (4.30 ϵ) for each ϵ , then from the remark after Theorem 4.1, we can take $\mathcal{S}_\epsilon = \mathcal{S}_0$ and hence $\Lambda_\epsilon = \Lambda_0$. \square

The system (4.36 ϵ) is called a *Fenichel normal form* for (4.30 ϵ), and the variables (a, b, c) are called *Fenichel coordinates*.

4.2. Silnikov Boundary Value Problem. We have seen in Section 4.1 that fast-slow systems (4.30 ϵ) can locally be converted into normal forms (4.36 ϵ), where A^u and A^s satisfy the gap condition (4.37), and E is a small term satisfying (4.38). If we append the system with the equation $\dot{\epsilon} = 0$ and then replace c by $\tilde{c} = (c, \epsilon)$, we obtain a system of the form

$$(4.39) \quad \begin{aligned} \dot{a} &= A^u(a, b, \tilde{c})a \\ \dot{b} &= A^s(a, b, \tilde{c})b \\ \dot{\tilde{c}} &= \tilde{h}(\tilde{c}) + E(a, b, \tilde{c}), \end{aligned}$$

for which (4.37) and (4.38) are satisfied with E replaced by \tilde{E} . For convenience, we will drop the tilde notation in (4.39) in the remaining discussion.

A Silnikov problem is the system (4.39) along with boundary data of the form

$$(4.40) \quad (b, c)(0) = (b^0, c^0), \quad a(T) = a^1,$$

where $T \geq 0$. This boundary value problem was posed in [Sil67] to study homoclinic bifurcation. A heuristic reason for the existence of solutions of a Silnikov problem is illustrated in Fig 3. Consider the simple case $\dot{a} = a$, $\dot{b} = -b$ and $\dot{c} = 0$. There are infinitely many trajectories contained in the box $\{0 \leq a \leq a^1, 0 \leq b \leq b^0\}$. We may parametrize the set of trajectories in $T \geq 0$ by $b(0) = b^0$ and $a(T) = a^1$. On the a -axis and b -axis, the trajectories tend to the origin in backward and forward time, respectively. This suggests that trajectories near the axes can stay for an arbitrarily long time in the box, which implies that for any large T there exists a trajectory satisfying $b(0) = b^0$ and $a(T) = a^1$. When T grows to infinity, the trajectories approach the axes. In the general case $\dot{a} = A^u a$ and $\dot{b} = A^s b$ in arbitrary dimension, both a - and b -spaces consist of solutions tending to the origin in forward or backward time, so we have the same conclusion.

The critical manifold for (4.39) is $\{a = 0, b = 0\}$, on which the system is governed by the limiting slow system

$$(4.41) \quad \dot{c} = h(c).$$

For a solution $(a(t), b(t), c(t))$ to the Silnikov boundary value problem (4.39) and (4.40), from conditions (4.37) and (4.38), it is natural to expect that $a(t)$ and $b(t)$ decay to 0 in backward time and forward time, respectively, and that $c(t)$ is approximately the solution of (4.41). A theorem from [Sch08b] asserts that this is the case:

Theorem 4.4 (Generalized Deng's Lemma [Sch08b]). *Consider the system (4.39) satisfying (4.37) and (4.38) with C^r coefficients, $r \geq 1$, defined on the closure of a bounded open set $B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$, where $B_{k,\Delta} = \{a \in \mathbb{R}^k : |a| < \Delta\}$, $\Delta > 0$, and V is a bounded open set in \mathbb{R}^l .*

Let K_0 and K_1 be compact subsets of V such that $K_0 \subset \text{Int}(K_1)$. For each $c^0 \in K_0$ let J_{c^0} be the maximal interval such that $\phi(t, c^0) \in \text{Int}(K_1)$ for all $t \in J_{c^0}$, where $\phi(t, c^0)$ is the solution of (4.41) with initial value c^0 . Let $\nu > 0$ be the number in (4.37). Suppose there exists $\beta > 0$ such that $\tilde{\nu} := \nu - r\beta > 0$ and

$$|\phi(t, c^0)| \leq Me^{\beta|t|} \quad \forall t \in J_{c^0}.$$

Then there is a number $\delta_0 > 0$ such that if $|a^1| < \delta_0$, $|b^0| < \delta_0$, $c^0 \in V_0$, and $T > 0$ is in J_{c^0} , then the Silnikov boundary value problem (4.39) and (4.40) has a solution $(a, b, c)(t, T, a^1, b^0, c^0)$ on the interval $0 \leq t \leq T$. Moreover, there is a number $K > 0$ such that for all (t, T, a^1, b^0, c^0) as above and for all multi-indices \mathbf{i} with $|\mathbf{i}| \leq r$,

$$(4.42) \quad \begin{aligned} |D_{\mathbf{i}}a(t, T, a^1, b^0, c^0)| &\leq Ke^{-\tilde{\nu}(T-t)} \\ |D_{\mathbf{i}}b(t, T, a^1, b^0, c^0)| &\leq Ke^{-\tilde{\nu}t} \\ |D_{\mathbf{i}}c(t, T, a^1, b^0, c^0) - D_{\mathbf{i}}\phi(t, c^0)| &\leq Ke^{-\tilde{\nu}T}. \end{aligned}$$

Sketch of Proof. Here we sketch the proof in [Sch08a]. Write (4.39) as

$$\begin{aligned} \dot{a} &= \tilde{A}^u(t, c^0)a + f(t, c^0, a, b, z) \\ \dot{b} &= \tilde{A}^s(t, c^0)b + g(t, c^0, a, b, z) \\ \dot{z} &= \tilde{A}^c(t, c^0)z + \theta(t, c^0, z) + \tilde{E}(t, c^0, a, b, z), \end{aligned}$$

where

$$\begin{aligned} \tilde{A}^i(t, c^0) &= A^i(0, 0, \phi(t, c^0)), \quad i = u, s, \\ \tilde{A}^c(t, c^0) &= Dh|_{\phi(t, c^0)} \end{aligned}$$

and

$$\tilde{E}(t, c^0, a, b, z) = E(a, b, \phi(t, c^0) + z).$$

Let $\Phi^i(t, s, c^0)$ be the solution operator for $\tilde{A}^i(t, c^0)$, $i = u, s, c$. Then $(a(t), b(t), c(t))$ is a solution of Silnikov problem (4.39) and (4.40) if and only if $c(t) = \phi(t, c^0) + z(t)$ and $\eta(t) = (a(t), b(t), z(t))$ satisfies

$$(4.43) \quad \begin{aligned} a(t) &= \Phi^u(t, T, c^0)a^1 - \int_t^T \Phi^u(t, s, c^0)f(s, c^0, \eta(s)) ds \\ b(t) &= \Phi^s(t, 0, c^0)b^0 + \int_0^t \Phi^s(t, s, c^0)g(s, c^0, \eta(s)) ds \\ z(t) &= \int_0^t \Phi^c(t, s, c^0)(\theta(s, c^0, z(s)) + \tilde{E}(s, c^0, \eta(s))) ds. \end{aligned}$$

Define an linear operator \mathcal{L} by the right-hand side of (4.43) for functions $\eta(t) = (a(t), b(t), z(t))$. It can be shown that the restriction of \mathcal{L} on a neighborhood of 0 in the space of functions $\eta(t) = (a(t), b(t), z(t))$ equipped with the norm

$$\|\eta\|_j = \sup_{0 \leq t \leq T} (e^{\tilde{\nu}(T-t)}|a(t)| + e^{\tilde{\nu}t}|b(t)| + e^{\tilde{\nu}T}|z(t)|)$$

is a contraction mapping. Hence the existence of solution of (4.39) and (4.40) follows from the standard Banach fixed point theorem. \square

Remark 3. Theorem 4.4 is a generalization of the *Strong λ -Lemma* in Deng [Den90], and *C^r -Inclination Theorem* in Brunovsky [Bru99]. In Deng's work, the boundary data lie near an equilibrium that may nonhyperbolic. In Brunovsky's work, the boundary data lie near a solution of a rectifiable slow flow on a

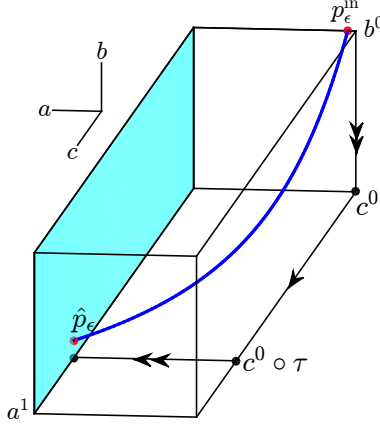


FIGURE 4. The entry point $(a_\epsilon^{\text{in}}, b^0, c^0)$ is close to $(0, b^0, c^0)$, and the exit point $(a^1, \hat{b}_\epsilon, \hat{c}_\epsilon)$ is close to $(a^1, 0, c^0 \circ \tau)$, as $\epsilon \rightarrow 0$.

normally hyperbolic invariant manifold. Schechter's work allows considering more general flows on normally hyperbolic invariant manifolds.

4.3. The Exchange Lemma. Consider (4.36 ϵ) as a special case of (4.39), and recall that (4.36 ϵ) is the normal form of fast-slow systems (4.30 ϵ). We will use Theorem 4.4 to analyze Silnikov problems for fast-slow systems (4.30 ϵ). The result turns out to be a variation of the $(k + \sigma)$ -Exchange Lemma [JT09, Tin94].

The Silnikov problem for (4.36 ϵ) corresponds to the boundary data

$$(4.44) \quad a(\tau/\epsilon) = a^1, \quad (b, c)(0) = (b^0, c^0),$$

with given $(a^1, b^0, c^0) \in \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$ and $\tau > 0$. It can be interpreted as finding trajectories for (4.36 ϵ) connecting the sets $\{b = b^0, c = c^0\}$ and $\{a = a^1\}$, with prescribed time interval $0 \leq t \leq \tau/\epsilon$; see Fig 4. Note that the set $\{b = b^0, c = c^0\}$ is of dimension k . The Exchange Lemma is a tool tracking the $(k + 1)$ -manifold \mathcal{I}_ϵ^* that evolves from a k -manifold \mathcal{I}_ϵ which is transverse to the center-stable manifold $\{a = 0\}$. The theory of Exchange Lemma was first developed in [JKL91, JK94, JKK96] to study singularly perturbed systems near a normally hyperbolic, locally invariant manifold. Some generalizations of the Exchange Lemma for a broader class of systems were given by W. Liu [Liu00] and Schechter [Sch08b].

Another generalization, given by Tin [Tin94], is the $(k + \sigma)$ -Exchange Lemma, $1 \leq \sigma \leq l$, which tracks the $(k + \sigma)$ -manifold \mathcal{I}_ϵ^* which evolves from a $(k + \sigma - 1)$ -manifold $\mathcal{I}_\epsilon = \{b = b^0, c^0 \in \Lambda\}$, where Λ is a $(\sigma - 1)$ -manifold. A major difference between the $(k + \sigma)$ -Exchange Lemma and the general Exchange Lemma in [Sch08b] is that the estimates (4.42) for the derivatives in slow variables were not considered in [Sch08b].

We analyze Silnikov problems for fast-slow systems in normal form (4.36 ϵ) in Lemma 4.1, and then, in Theorem 4.5, return to (4.30 ϵ) to present a version of the $(k + \sigma)$ -Exchange Lemma.

Lemma 4.1. *Consider a system of the form (4.36 ϵ) satisfying (4.37) and (4.38) defined on the closure of a bounded open set $B_{k,\Delta} \times B_{m,\Delta} \times V \subset \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^l$, where the coefficients are C^r for some integer $r \geq 0$. Let $\Lambda \subset V$ be a $(\sigma - 1)$ -dimensional manifold, $1 \leq \sigma \leq l$ and $\tau_0 > 0$. Suppose*

$$c \circ [0, \tau_0] \subset V \quad \forall c \in \Lambda,$$

where \circ denotes the flow for the limiting slow system (4.33). Let $J \subset (0, \tau_0)$ be a closed interval and $\mathcal{A} \subset B_{k,\Delta} \setminus \{0\}$ be a compact set. Then for each small $\epsilon > 0$ and $(a^1, c^0, \tau) \in \mathcal{A} \times \Lambda \times J$, the boundary value problem (4.30 ϵ) and (4.44) has a unique solution, denoted by $(a, b, c)(t; \tau, a^1, b^0, c^0, \epsilon)$, $t \in [0, \tau/\epsilon]$. Moreover, if we set

$$(4.45) \quad p_\epsilon = (a, b, c)(0; \tau, a^1, b^0, c^0, \epsilon), \quad q_\epsilon = (a, b, c)(\tau/\epsilon; \tau, a^1, b^0, c^0, \epsilon),$$

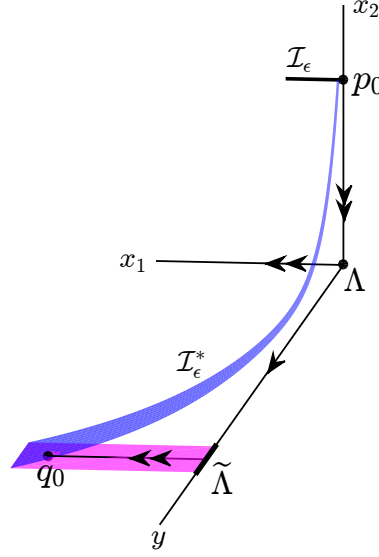


FIGURE 5. The $(k + \sigma)$ -Exchange Lemma asserts that \mathcal{I}_ϵ^* is C^1 -close to $W_0^u(\tilde{\Lambda})$ in a neighborhood of q_0 .

then

$$(4.46) \quad \|p_\epsilon - (0, b^0, c^0)\|_{C^r(\mathcal{A} \times \Lambda \times J)} + \|q_\epsilon - (a^1, 0, c^0 \circ \tau)\|_{C^r(\mathcal{A} \times \Lambda \times J)} \leq C e^{-\tilde{\nu}/\epsilon}$$

for some positive constants \tilde{C} and $\tilde{\nu}$. See Fig 4.

Sketch of Proof. Existence of solutions follows directly from Theorem 4.4, so it remains to prove (4.46). Write $p_\epsilon = (a_\epsilon^{\text{in}}, b^0, c^0)$ and $q_\epsilon = (a^1, \hat{b}_\epsilon, \hat{c}_\epsilon)$, then (4.46) is equivalent to

$$(4.47) \quad \|(a_\epsilon^{\text{in}}, \hat{b}_\epsilon, \hat{c}_\epsilon - c^0 \circ \tau)\|_{C^r(\mathcal{A} \times \Lambda \times J)} \leq \tilde{C} e^{-\tilde{\nu}/\epsilon}.$$

The estimate of the derivatives in $(a^1, c^0) \in \mathcal{A} \times \Lambda$ in (4.47) follows directly from (4.42). To prove the estimate of the derivatives in $\tau \in J$, note that from (4.43) we have

$$(4.48) \quad \begin{aligned} a_\epsilon^{\text{in}} &= \Phi^u(0, \tau/\epsilon, c^0) a^1 - \int_0^{\tau/\epsilon} \Phi^u(0, s, c^0) f(s, c^0, \eta(s)) ds \\ \hat{b}_\epsilon &= \Phi^s(\tau/\epsilon, 0, c^0) b^0 + \int_0^{\tau/\epsilon} \Phi^s(\tau/\epsilon, s, c^0) g(s, c^0, \eta(s)) ds \\ \hat{c}_\epsilon &= c^0 \circ \tau + \int_0^{\tau/\epsilon} \Phi^c(\tau/\epsilon, s, c^0) (\theta(s, c^0, z(s)) + \tilde{E}(s, c^0, \eta(s))) ds. \end{aligned}$$

As in the proof of Theorem 4.4, it can be shown that the derivatives of the integrands in (4.48) are exponentially small, so we obtain (4.47). \square

The following theorem is a modification of the $(k + \sigma)$ -Exchange Lemma. The main difference is that in this version we assert the existence of certain trajectories, while in the original version those trajectories were assumed to exist. The proof of the original theorem [Tin94] is based on tracking tangent spaces to an invariant manifold using linearized differential equations in terms of differential forms, while the approach we present below relies on estimates for solution operators, following closely to the proof of the general Exchange Lemma in [Sch08a].

Theorem 4.5. Consider a system of the form (4.30 ϵ) where $(x, y) \in \mathbb{R}^n \times \mathbb{R}^l$, and f and g are C^r functions for some $r \geq 2$. Let \mathcal{S}_0 be a normally hyperbolic critical manifold for (4.31), and suppose $D_x f|_{\mathcal{S}_0}$ has a

splitting of k unstable eigenvalues and m stable eigenvalues, $k + m = n$. Let $\bar{q}_0 \in W_0^u(\mathcal{S}_0) \setminus \mathcal{S}_0$, $\bar{p}_0 \in W_0^s(\mathcal{S}_0) \setminus \mathcal{S}_0$, $\bar{\tau}_0 > 0$, and assume

$$(4.49) \quad \pi^s(\bar{p}_0) \circ [0, \bar{\tau}_0] \subset \mathcal{S}_0 \quad \text{and} \quad \pi^u(\bar{q}_0) = \pi^s(\bar{p}_0) \circ \tau_0,$$

where \circ denotes the flow for the limiting slow system (4.33), and $\pi^{s,u}$ are the projections into \mathcal{S}_0 along stable/unstable fibers with respect to the limiting fast system (4.31). Let $\{\mathcal{I}_\epsilon\}_{\epsilon \in [0, \epsilon_0]}$ be a C^r family of $(k + \sigma - 1)$ -dimensional manifolds, $1 \leq \sigma \leq l$, and suppose

(T1) \mathcal{I}_0 is transverse to $W^s(\mathcal{S}_0)$ at p_0 , and $\Lambda := \pi^s(\mathcal{I}_0 \cap W^s(\mathcal{S}_0))$ is of dimension $(\sigma - 1)$.

(T2) the slow flow (4.33) is not tangent to Λ at $\pi^s(\bar{p}_0)$.

(T3) The trajectory $\pi^s(p_0) \circ [0, \tau_0]$ is rectifiable and not self-intersecting.

Let

$$(4.50) \quad \mathcal{I}_\epsilon^* = \mathcal{I}_\epsilon \cdot [0, \infty),$$

where \cdot denotes the flow for (4.30 ϵ). Choose a compact interval $J \subset (0, \infty)$ containing $\bar{\tau}_0$ satisfying $\Lambda \circ J \subset \mathcal{S}_0$, and set $\tilde{\Lambda} = \Lambda \circ J$. Then there exists a neighborhood V_0 of \bar{q}_0 such that

$$(4.51) \quad \mathcal{I}_\epsilon^* \cap V_0 \text{ is } C^{r-2} \text{ } O(\epsilon)\text{-close to } W_0^u(\tilde{\Lambda}) \cap V_0.$$

See Fig 5. Moreover, given any sequence $\bar{q}_\epsilon \in \mathcal{I}_\epsilon^* \cap V_0$ such that $\bar{q}_\epsilon \rightarrow \bar{q}_0$, there exists a sequence $(\bar{p}_\epsilon, \bar{\tau}_\epsilon) \in \mathcal{I}_\epsilon \times J$ which converges to $(\bar{p}_0, \bar{\tau}_0)$ and satisfies that, setting $T_\epsilon = \bar{\tau}_\epsilon/\epsilon$,

$$(4.52) \quad \bar{q}_\epsilon = \bar{p}_\epsilon \cdot T_\epsilon \quad \forall \epsilon > 0,$$

and

$$(4.53) \quad T_\epsilon = (\tau_0 + o(1))\epsilon^{-1}.$$

Proof. Under the assumption (??), from [Den90, Lemma 2.2], after a C^{r-2} change of coordinates, we can convert (4.30 ϵ) to (4.36 ϵ), and, from (T1), we may assume

$$(4.54) \quad \mathcal{I}_\epsilon = B_{k,\Delta} \times \{b^0\} \times \Lambda$$

for some constant $b^0 \in B_{m,\Delta} \setminus \{0\}$.

Since $\bar{q}_0 \in W_0^u(\mathcal{S}_0) \setminus \mathcal{S}_0$, we have $a(\bar{q}_0) \neq 0$ and $b(\bar{q}_0) = 0$, where $a(\bar{q}_0)$ and $b(\bar{q}_0)$ denote the a - and b -coordinates of \bar{q}_0 . Set

$$(4.55) \quad \mathcal{A} = \{a \in \mathbb{R}^k : |a - a(\bar{q}_0)| < \Delta_1\}$$

for some positive number $\Delta_1 < \frac{1}{2} \min\{\Delta, |a(\bar{q}_0)|\}$, so that $\mathcal{A} \subset B_{k,\Delta} \setminus \{0\}$. Let p_ϵ and q_ϵ be the functions of $(a^1, c^0, \tau) \in \mathcal{A} \times \Lambda \times J$ defined by (4.45). From (4.54) we see that (p_ϵ, τ) parametrizes $\mathcal{I}_\epsilon \times J$ in a neighborhood of (p_0, τ_0) . Hence q_ϵ parametrizes \mathcal{I}_ϵ^* in neighborhoods of \bar{q}_0 . The estimate (4.46) holds with r replaced by $r - 2$. In particular,

$$\|q_\epsilon - (a^1, 0, c^0 \circ \tau)\|_{C^{r-2}(\mathcal{A} \times \Lambda \times J)} \leq C e^{-\bar{\nu}/\epsilon}.$$

Note that

$$(4.56) \quad \begin{aligned} W^u(\tilde{\Lambda}) &= \{(a, b, c) : b = 0, c \in \tilde{\Lambda}\} \\ &= \{(a, b, c^0 \circ \tau) : b = 0, c^0 \in \Lambda, \tau \in J\}, \end{aligned}$$

so we obtain (4.51).

Next we consider the sequence $\bar{q}_\epsilon \in \mathcal{I}_\epsilon$ given in the statement. Choose $(a_\epsilon^1, c_\epsilon^0, \tau_\epsilon) \in \mathcal{A} \times \Lambda \times J$ such that $\bar{q}_\epsilon = q_\epsilon(a_\epsilon^1, c_\epsilon^0, \tau_\epsilon)$, and set $\bar{p}_\epsilon = p_\epsilon(a_\epsilon^1, c_\epsilon^0, \tau_\epsilon)$. Then by definition $\bar{q}_\epsilon = \bar{p}_\epsilon \cdot (\bar{\tau}_\epsilon/\epsilon)$. From (T2) and (T3), $\tilde{\Lambda}$ is a σ -dimensional manifold, and for any $c^1 \in \tilde{\Lambda}$, there exists unique $(c^0, \tau_0) \in \Lambda \times J$ such that $c^1 = c^0 \circ \tau_0$. Hence (4.52) uniquely determines $\bar{p}_0 \in \mathcal{I}_0 \cap W^s(\Lambda)$ and $\bar{\tau}_0 \in J$. To show $p_\epsilon \rightarrow \bar{p}_0$ and $\tau_\epsilon \rightarrow \bar{\tau}_0$, since $(p_\epsilon, \tau_\epsilon)$ lies in the compact set $\Lambda \times J$, it suffices to show that every convergent subsequence of $\{(c_\epsilon, \tau_\epsilon)\}$ converges to $(\bar{c}^0, \bar{\tau}_0)$. Note that from the equation for \hat{c}_ϵ in (4.48), we have

$$(4.57) \quad \hat{c}_\epsilon = c_\epsilon^0 \circ \tau_\epsilon + o(1).$$

Since $q_\epsilon \rightarrow \bar{q}_0 \equiv (\bar{a}^1, 0, \bar{c}^1)$, given any convergent subsequence $(c_{\epsilon_j}, \tau_{\epsilon_j})$ of $(c_\epsilon, \tau_\epsilon)$, say $(c_{\epsilon_j}, \tau_{\epsilon_j}) \rightarrow (\bar{c}^0, \bar{\tau}_0)$, from (4.57) we obtain $\bar{c}^1 = \bar{c}^0 \circ \bar{\tau}_0$. From (4.49) we have $\bar{c}^1 = \bar{c}^0 \circ \bar{\tau}_0$. Hence $(\bar{c}^0, \bar{\tau}_0) = (\bar{c}^0, \bar{\tau}_0)$. This completes the proof. \square

5. SINGULAR CONFIGURATION

The fast-slow system (2.12 ϵ) has multiple limiting subsystems corresponding to different time scales. In this section we will find trajectories, called *singular trajectories*, for those subsystems such that the union of those trajectories joins the end states u_L and u_R . The union of those singular trajectories is called a *singular configuration*. In later sections we will show that there are solutions of (2.12 ϵ) close to the singular configuration.

5.1. End States \mathcal{U}_L and \mathcal{U}_R . The system (2.15) has a normally hyperbolic critical manifold

$$(5.58) \quad \mathcal{S}_0 = \{(u, w, \xi) : f(u) - \xi u - w = 0, \xi \neq \operatorname{Re}(\lambda_{\pm}(u))\},$$

where $\lambda_{\pm}(u)$ are the eigenvalues of $Df(u)$, defined in (2.16). The limiting slow system for (2.14 ϵ) is

$$(5.59) \quad \begin{aligned} 0 &= f(u) - \xi u - w \\ w' &= -u \\ \xi' &= 1. \end{aligned}$$

From (H1) we have $s < \operatorname{Re}(\lambda_{\pm}(u_L))$, so $(u_L, w_L, s) \in \mathcal{S}_0$. Choose $\delta > 0$ so that $s + 2\delta < \operatorname{Re}(\lambda_{\pm}(u_L))$, and set

$$(5.60) \quad \begin{aligned} \mathcal{U}_L &= (u_L, w_L, s) \underset{(5.59)}{\bullet} (-\infty, \delta] \\ &= \{(u, w, \xi) : u = u_L, w = w_L - \alpha_1 u_L, \xi = s + \alpha_1, \alpha_1 \in (-\infty, \delta]\}, \end{aligned}$$

where $\underset{(5.59)}{\bullet}$ denotes the flow for (5.59). It is clear that $\mathcal{U}_L \subset \mathcal{S}_0$ is normally hyperbolic with respect to (2.15), and is locally invariant with respect to (2.12 ϵ).

Note that each point in \mathcal{U}_L is a hyperbolic equilibrium for the 2-dimensional system (2.13), and the unstable manifold $W_0^u(\mathcal{U}_L)$ is naturally defined.

Proposition 5.1. *Assume (H1). Let \mathcal{U}_L be defined by (5.60). Fix any $k \geq 1$. There exists a family of invariant manifolds $W_{\epsilon}^u(\mathcal{U}_L)$ which are C^k $O(\epsilon)$ -close to $W_0^u(\mathcal{U}_L)$ such that for any continuous family $\{\mathcal{I}_{\epsilon}\}_{\epsilon \in [0, \epsilon_0]}$ of compact sets $\mathcal{I}_{\epsilon} \subset W_{\epsilon}^u(\mathcal{U}_L)$,*

$$(5.61) \quad \operatorname{dist}(p \underset{(2.12\epsilon)}{\bullet} t, \mathcal{U}_L) \leq C e^{\mu t} \quad \forall p \in \mathcal{I}_{\epsilon}, t \leq 0, \epsilon \in [0, \epsilon_0],$$

for some positive constants C and μ .

Proof. This follows from Theorem 4.3 by taking \mathcal{U}_L to be \mathcal{U}_0 . Although \mathcal{U}_L is not compact, it is uniformly normally hyperbolic since $\xi - \operatorname{Re}(\lambda_{\pm}(u_L)) < -\delta$ on \mathcal{U}_L , and the proof of Theorem 4.3 in [Jon95, Theorem 4] is still valid. \square

Remark 4. Proposition 5.1 was also asserted in [Sch04, Liu04, KT12].

From (H1) we also have, by decreasing δ if necessary, $s - 2\delta > \operatorname{Re}(\lambda_{\pm}(u_R))$, and hence a similar result holds for the set \mathcal{U}_R defined by

$$(5.62) \quad \begin{aligned} \mathcal{U}_R &= (u_R, w_R, s) \underset{(5.59)}{\bullet} [-\delta, \infty) \\ &= \{(u, w, \xi) : u = u_R, w = w_R - \alpha_2 u_R, \xi = s + \alpha_2, \alpha_2 \in [-\delta, \infty)\}. \end{aligned}$$

Proposition 5.2. *Assume (H1). Let \mathcal{U}_R be defined by (5.62). Fix any $k \geq 1$. There exists a family of invariant manifolds $W_{\epsilon}^s(\mathcal{U}_R)$ which are C^k $O(\epsilon)$ -close to $W_0^s(\mathcal{U}_R)$ such that for any continuous family $\{\mathcal{J}_{\epsilon}\}_{\epsilon \in [0, \epsilon_0]}$ of compact sets $\mathcal{J}_{\epsilon} \subset W_{\epsilon}^s(\mathcal{U}_R)$,*

$$(5.63) \quad \operatorname{dist}(p \underset{(2.12\epsilon)}{\bullet} t, \mathcal{U}_R) \leq C e^{-\mu t} \quad \forall p \in \mathcal{J}_{\epsilon}, t \geq 0, \epsilon \in [0, \epsilon_0],$$

for some positive constants C and μ .

5.2. Intermediate States \mathcal{P}_L and \mathcal{P}_R . Consider the system (2.14 ϵ). In order to study the dynamics at $\{v = +\infty\}$, we set $r = 1/v$ and $\kappa = \epsilon \log(1/r)$. Then (2.14 ϵ) is converted, after multiplying the equations by r , to

$$\begin{aligned}
 (5.64\epsilon) \quad & \dot{\beta} = B_1(\beta) - \xi\beta r - w_1 r \\
 & \dot{r} = -rB_2(\beta) + \xi r^2 + w_2 r^3 \\
 & \dot{w}_1 = -\epsilon\beta r \\
 & \dot{w}_2 = -\epsilon \\
 & \dot{\xi} = \epsilon r \\
 & \dot{\kappa} = \epsilon(B_2(\beta) + \xi r + w_2 r^2).
 \end{aligned}$$

Note that the time variable in (5.64 ϵ) is different from that of (2.14 ϵ). We use the same dot symbol to denote derivatives, but there should be no ambiguity since the different time scales can be distinguished by comparing the term $\dot{\xi}$.

The limiting fast system for (5.64 ϵ) is

$$\begin{aligned}
 (5.65) \quad & \dot{\beta} = B_1(\beta) - \xi\beta r - w_1 r \\
 & \dot{r} = -rB_2(\beta) + \xi r^2 + w_2 r^3 \\
 & \dot{w}_1 = 0, \quad \dot{w}_2 = 0, \quad \dot{\xi} = 0, \quad \dot{\kappa} = 0.
 \end{aligned}$$

The obvious equilibria for (5.65), besides $(\beta_L, r_L, w_{1L}, w_{2L}, s)$ and $(\beta_R, r_R, w_{1L}, w_{2L}, s)$, where $r_L = 1/v_L$ and $r_R = 1/v_R$, are

$$(5.66) \quad \mathcal{P}_L = \{(\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_1, r = 0\},$$

$$(5.67) \quad \mathcal{P}_R = \{(\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_2, r = 0\}.$$

The limiting slow system on \mathcal{P}_L is

$$(5.68) \quad w'_1 = 0, \quad w'_2 = -1, \quad \xi' = 0, \quad \kappa' = B_2(\rho_1),$$

and on \mathcal{P}_R is

$$(5.69) \quad w'_1 = 0, \quad w'_2 = -1, \quad \xi' = 0, \quad \kappa' = B_2(\rho_2)$$

The Fenichel coordinates near \mathcal{P}_L can be described as follows.

Proposition 5.3. *Let $W_{\epsilon}^{u,s}(\mathcal{P}_L)$ be the C^k unstable/stable manifolds of \mathcal{P}_L for (5.64 ϵ), $k \geq 1$. Then there exists a C^k function $\hat{\beta} = \hat{\beta}(\beta, r, w_1, w_2, \xi, \kappa, \epsilon)$ such that*

$$(5.70) \quad \hat{\beta} = \beta \quad \text{when } r = 0$$

and $(\hat{\beta}, r, w_1, w_2, \xi, \kappa)$ is a change of coordinates near \mathcal{P}_L satisfying

$$(5.71) \quad W_{\epsilon}^s(\mathcal{P}_L) = \{(\hat{\beta}, r, w_1, w_2, \xi, \kappa) : \hat{\beta} = \rho_1\}$$

$$(5.72) \quad W_{\epsilon}^u(\mathcal{P}_L) = \{(\hat{\beta}, r, w_1, w_2, \xi, \kappa) : r = 0\}.$$

Moreover, the projection $\pi_{\epsilon, \mathcal{P}_L}^s$ into \mathcal{P}_L along stable fibers with respect to (5.64 ϵ) is

$$(5.73) \quad \pi_{\epsilon, \mathcal{P}_L}^s(\rho_1, r, w_1, w_2, \xi, \kappa) = (\rho_1, 0, w_1, w_2, \xi, \kappa)$$

in $(\hat{\beta}, r, w_1, w_2, \xi, \kappa)$ -coordinates.

Proof. The linearization of (5.65) at \mathcal{P}_L corresponds to the matrix

$$(5.74) \quad \begin{pmatrix} B'_1(\rho_1) & -\xi\rho_1 \\ 0 & -B_2(\rho_1) \end{pmatrix} = \begin{pmatrix} 1 - \frac{\rho_2}{\rho_1} & -\xi\rho_1 \\ 0 & -\frac{1}{2}(1 - \frac{\rho_2}{\rho_1}) \end{pmatrix}$$

which has one positive and one negative eigenvalue. Note that \mathcal{P}_L is invariant under (5.64 ϵ) for each ϵ . From Theorem 4.1 and the remark following it, $W_{\epsilon}^s(\mathcal{P}_L)$ and $W_{\epsilon}^u(\mathcal{P}_L)$ are well defined and both have dimension 1, and we may take $W_{\epsilon}^u(\mathcal{P}_L) = \{r = 0\}$. Note that $\{\beta = \rho_1\}$ is transverse to $W_{\epsilon}^s(\mathcal{P}_L)$, so we can choose Fenichel coordinates (a, b, c) corresponding to this splitting with $b = r$ and

$$a = \beta - \rho_1 + \phi(w_1, w_2, \xi, \kappa, \epsilon, r)r$$

for some C^k function ϕ . Let $\hat{\beta} = a + \rho_1$. Then the desired result follows. \square

An analogous result holds for \mathcal{P}_R . We omit it here.

5.3. Transversal Intersections. Fix small $r^0 > 0$ such that γ_1 intersects $\{r = r^0\}$ at a unique point. Denote this point by p_0^{in} . That is,

$$(5.75) \quad p_0^{\text{in}} = \gamma_1 \cap \{r = r^0\}.$$

We set

$$(5.76) \quad \mathcal{I}_\epsilon = W_\epsilon^u(\mathcal{U}_L) \cap \{r = r^0\} \cap V_1,$$

where V_1 is an open neighborhood of p_0^{in} such that \mathcal{I}_ϵ can be parametrized as

$$(5.77) \quad \begin{aligned} \mathcal{I}_\epsilon &= \{(\hat{\beta}, r, w_1, w_2, \xi, \kappa) : r = r^0, \kappa = \epsilon \log(1/r^0), \\ (w_1, w_2, \xi) &= (w_{1L}, w_{2L}, s) + \alpha_1(-\beta_L, -v_L, 1) + \epsilon\theta(a, \alpha_1, \epsilon), \\ |a| < \Delta_1, |\alpha_1| < \Delta_1\}, \end{aligned}$$

where the coordinates $(\hat{\beta}, r, w_1, w_2, \xi, \kappa)$ are defined in Proposition 5.3. From (5.71) we see that \mathcal{I}_0 and $W_0^s(\mathcal{P}_L)$ intersect transversally at p_0^{in} , and if we set

$$(5.78) \quad \Lambda_L = \pi_{0, \mathcal{P}_L}^s(\mathcal{I}_0 \cap W_0^s(\mathcal{P}_L)),$$

where π_{0, \mathcal{P}_L}^s is the projection into \mathcal{P}_L along stable fibers with respect to (5.65), then

$$(5.79) \quad \begin{aligned} \Lambda_L &= \{(\beta, r, w_1, w_2, \xi, \kappa) : \beta = \rho_1, r = 0, \kappa = 0, \\ (w_1, w_2, \xi) &= (w_{1L}, w_{2L}, s) + \alpha_1(-\beta_L, -v_L, 1), \\ |\alpha_1| < \Delta_1\}. \end{aligned}$$

Similarly, by shrinking r^0 if necessary, γ_2 intersects $\{r = r^0\}$ at a unique point

$$(5.80) \quad p_0^{\text{out}} = \gamma_2 \cap \{r = r^0\}.$$

Set

$$(5.81) \quad \mathcal{J}_\epsilon = W_\epsilon^s(\mathcal{U}_R) \cap \{r = r^0\} \cap V_2,$$

where V_2 is an open neighborhood of p_0^{out} such that \mathcal{J}_ϵ has a parametrization analogous to (5.77). Then \mathcal{J}_0 is transverse to $W_0^u(\mathcal{P}_R)$ at p_0^{out} , and we set

$$(5.82) \quad \Lambda_R = \pi_{0, \mathcal{P}_R}^u(\mathcal{J}_0 \cap W_0^u(\mathcal{P}_R)),$$

where π_{0, \mathcal{P}_R}^u is the projection into \mathcal{P}_R along unstable fibers with respect to (5.65).

To connect p_0^{in} and p_0^{out} , we have the following

Proposition 5.4. *The system (5.65) has a trajectory*

$$(5.83) \quad \gamma_0 = \{(\beta, 0, w_{1L}, w_{20}, s, \kappa_0) : \beta \in (\rho_2, \rho_1)\},$$

which joins the points

$$\pi_{\mathcal{P}_R}^s(p_0^{\text{out}}) \underset{(5.68)}{\bullet} \tau_{10} \in \mathcal{P}_L \quad \text{and} \quad \pi_{\mathcal{P}_L}^u(p_0^{\text{in}}) \underset{(5.69)}{\bullet} (-\tau_{20}) \in \mathcal{P}_R,$$

where

$$(5.84) \quad w_{20} = w_{2L} + \frac{\rho_1}{\rho_1 + \rho_2}(w_{2L} - w_{2R}), \quad \kappa_0 = \frac{\rho_1(\rho_1 - \rho_2)}{2\rho_2(\rho_1 + \rho_2)},$$

and

$$(5.85) \quad \tau_{10} = \frac{\rho_2}{\rho_1 + \rho_2}(w_{2L} - w_{2R}), \quad \tau_{20} = \frac{\rho_1}{\rho_1 + \rho_2}(w_{2L} - w_{2R}).$$

Moreover, if we set

$$(5.86) \quad \tilde{\Lambda}_L = \Lambda_L \underset{(5.68)}{\bullet} [\tau_{1-}, \tau_{1+}] \quad \text{and} \quad \tilde{\Lambda}_R = \Lambda_R \underset{(5.69)}{\bullet} (-[\tau_{2-}, \tau_{2+}]),$$

where $\tau_{1-} < \tau_{10} < \tau_{1+}$ and $\tau_{2-} < \tau_{20} < \tau_{2+}$, then $W_0^u(\tilde{\Lambda}_L)$ and $W_0^s(\tilde{\Lambda}_R)$ intersect transversally along γ_0 in the space $\{r = 0\}$.

Proof. Note that the restriction of the system (5.65) on $\{r = 0\}$ is simply $\dot{\beta} = B_1(\beta)$, so every trajectory of (5.65) joins $\{\beta = \rho_1\}$ and $\{\beta = \rho_2\}$. Also note that

$$\pi_{\mathcal{P}_R}^s(p_0^{\text{out}}) \underset{(5.68)}{\bullet} \tau = (\rho_1, 0, w_{1L}, w_{2L}, s, 0) + \tau(0, 0, -1, 0, B_2(\rho_1))$$

$$\pi_{\mathcal{P}_L}^u(p_0^{\text{out}}) \underset{(5.69)}{\bullet} \tau = (\rho_2, 0, w_{1R}, w_{2R}, s, 0) + \tau(0, 0, 0, -1, 0, B_2(\rho_2)), \quad \forall \tau \in \mathbb{R},$$

in $(\beta, r, w_1, w_2, \xi, \kappa)$ -coordinates. Hence γ_0 defined in (5.83) joins $\pi_{\mathcal{P}_R}^s(p_0^{\text{out}}) \underset{(5.68)}{\bullet} \tau_{10}$ and $\pi_{\mathcal{P}_L}^u(p_0^{\text{out}}) \underset{(5.69)}{\bullet} (-\tau_{20})$ if

$$(5.87) \quad w_{20} = w_{2L} - \tau_{10} = w_{2R} + \tau_{20}, \quad \kappa_0 = B_2(\rho_1)\tau_{10} = -B_2(\rho_2)\tau_{20},$$

which gives (5.84) and (5.85).

Let $\tilde{\Lambda}_L$ and $\tilde{\Lambda}_R$ be defined in (5.86). From the parameterizations (5.72) and (5.79), we have

$$(5.88) \quad \begin{aligned} W_0^u(\tilde{\Lambda}_L) &= \{(\beta, r, w_1, w_2, \xi, \kappa) : r = 0, \\ (w_1, w_2, \xi, \kappa) &= (w_{1L}, w_{2L}, s, 0) + \alpha_1(-\beta_L, -v_L, 1, 0) + \tau_1(0, -1, 0, B_2(\rho_1)), \\ \beta &\in (\rho_2, \rho_1), |\alpha_1| < \Delta_1, \tau_1 \in [\tau_{1-}, \tau_{1+}]\} \end{aligned}$$

and

$$(5.89) \quad \begin{aligned} W_0^s(\tilde{\Lambda}_R) &= \{(\beta, r, w_1, w_2, \xi, \kappa) : r = 0, \\ (w_1, w_2, \xi, \kappa) &= (w_{1R}, w_{2R}, s, 0) + \alpha_1(-\beta_R, -v_R, 1, 0) - \tau_2(0, -1, 0, B_2(\rho_2)), \\ \beta &\in (\rho_2, \rho_1), |\alpha_2| < \Delta_1, \tau_2 \in [\tau_{2-}, \tau_{2+}]\}. \end{aligned}$$

Fix any $q_0 \in \gamma_0$, we have

$$T_{q_0}W_0^u(\tilde{\Lambda}_L) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\beta_L \\ -v_L \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ B_2(\rho_1) \end{pmatrix} \right\}$$

and

$$T_{q_0}W_0^s(\tilde{\Lambda}_R) = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\beta_R \\ -v_R \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ B_2(\rho_2) \end{pmatrix} \right\}.$$

Hence $T_{q_0}W_0^u(\tilde{\Lambda}_L)$ and $T_{q_0}W_0^s(\tilde{\Lambda}_R)$ span the space $\{r = 0\}$. This means $W_0^u(\tilde{\Lambda}_L)$ and $W_0^s(\tilde{\Lambda}_R)$ intersect transversally in the space $\{r = 0\}$ at q_0 . \square

Let γ_0 be the trajectory defined in Proposition 5.4. We set

$$(5.90) \quad q_0 = \gamma_0 \cap \Gamma,$$

where

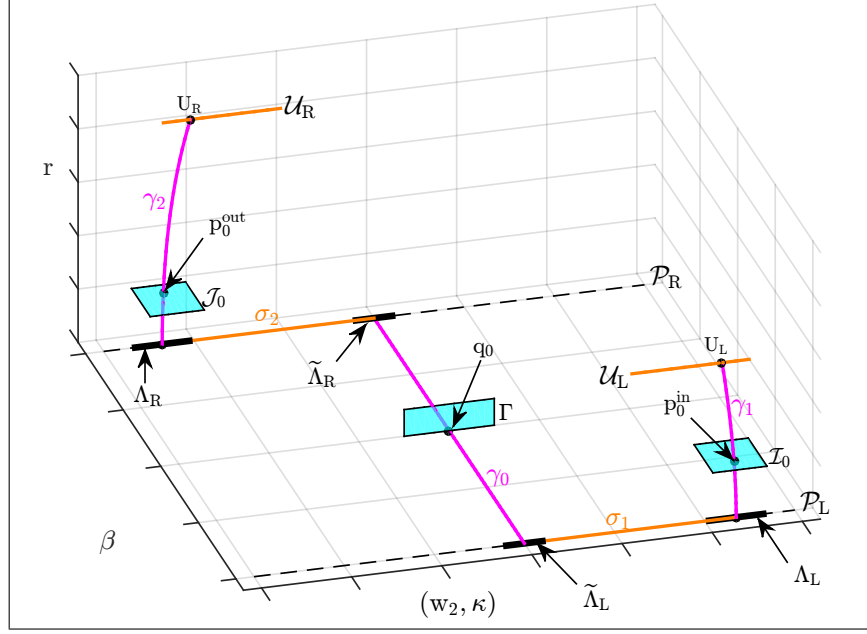
$$(5.91) \quad \Gamma = \{(\beta, r, w_1, w_2, \xi, \kappa) : \beta = \frac{\rho_1 + \rho_2}{2}\}.$$

Then

$$(5.92) \quad \pi_{\mathcal{P}_L}^u(q_0) = \pi_{\mathcal{P}_L}^s(p_0^{\text{in}}) \underset{(5.68)}{\bullet} \tau_{10}, \quad \pi_{\mathcal{P}_R}^s(q_0) = \pi_{\mathcal{P}_R}^u(p_0^{\text{out}}) \underset{(5.69)}{\bullet} (-\tau_{20}).$$

Let

$$(5.93) \quad \begin{aligned} \sigma_1 &= \pi_{\mathcal{P}_L}^s(p_0^{\text{in}}) \underset{(5.68)}{\bullet} [0, \tau_{10}] \\ &= \{(\rho_2, 0, w_{1L}, w_{2L} - \tau, s, B_2(\rho_1)\tau) : \tau \in [0, \tau_{10}]\} \end{aligned}$$


 FIGURE 6. The singular configuration $\gamma_1 \cup \sigma_1 \cup \gamma_0 \cup \sigma_2 \cup \gamma_2$ connects \mathcal{U}_L and \mathcal{U}_R .

and

$$\begin{aligned}
 \sigma_2 &= \pi_{\mathcal{P}_L}^u(p_0^{\text{in}}) \underset{(5.69)}{\bullet} [-\tau_{20}, 0] \\
 &= \{(\rho_1, 0, w_{1R}, w_{2R} + \tau, s, -B_2(\rho_2)\tau) : \tau \in [0, \tau_{20}]\}
 \end{aligned}
 \tag{5.94}$$

in $(\beta, r, w_1, w_2, \xi, \kappa)$ -coordinates. Then we obtain the singular configuration

$$\gamma_1 \cup \sigma_1 \cup \gamma_0 \cup \sigma_2 \cup \gamma_2
 \tag{5.95}$$

connecting \mathcal{U}_L and \mathcal{U}_R . See Fig 6.

6. COMPLETING THE PROOF OF THE MAIN THEOREM

We split the proof of the main theorem into two parts. In the first subsection we prove the existence of solutions of the boundary value problem (2.9 ϵ), (2.10), and show that (2.20) holds. In the second subsection we derive the weak limit (2.21).

6.1. Existence of Trajectories.

Proposition 6.1. *Assume (H1)-(H3). Let p_0^{in} , p_0^{out} , q_0 , \mathcal{I}_ϵ , \mathcal{J}_ϵ and Σ be defined in Section 5.3. Then for each small $\epsilon > 0$, there exist $p_\epsilon^{\text{in}} \in \mathcal{I}_\epsilon$, $p_\epsilon^{\text{out}} \in \mathcal{J}_\epsilon$, $q_\epsilon \in \Gamma$ and $T_{1\epsilon}, T_{2\epsilon} > 0$ such that*

$$p_\epsilon^{\text{in}} = q_\epsilon \cdot (-T_{1\epsilon}), \quad p_\epsilon^{\text{out}} = q_\epsilon \cdot T_{2\epsilon},
 \tag{6.96}$$

where \cdot denotes the flow of (5.64 ϵ), satisfying

$$(p_\epsilon^{\text{in}}, p_\epsilon^{\text{out}}, q_\epsilon) = (p_0^{\text{in}}, p_0^{\text{out}}, q_0) + o(1)
 \tag{6.97}$$

and

$$T_{1\epsilon} = (\tau_{10} + o(1))\epsilon^{-1}, \quad T_{2\epsilon} = (\tau_{20} + o(1))\epsilon^{-1},
 \tag{6.98}$$

as $\epsilon \rightarrow 0$, where τ_{10} and τ_{20} are defined in (5.85). Moreover, if we set $\kappa_\epsilon(\sigma)$ to be the κ -coordinate of $q_\epsilon \cdot \sigma$, $\sigma \in [-T_{1\epsilon}, T_{2\epsilon}]$, then

$$\max_{\sigma \in [-T_{1\epsilon}, T_{2\epsilon}]} \kappa_\epsilon(\sigma) = \kappa_0 + o(1),
 \tag{6.99}$$

where κ_0 is defined in (5.84).

Proof. We will apply the Exchange Lemma (Theorem 4.5) with $(k, m, l, \sigma) = (1, 1, 3, 1)$. From (5.77) and (5.71), we know that the $(k + \sigma)$ -manifold \mathcal{I}_0 is transverse to the $(m + l)$ -manifold $W_0^s(\mathcal{P}_L)$ at p_0^{in} , and the image of the projection

$$\pi_{\mathcal{P}_L}^s(\mathcal{I}_0 \cap W_0^s(\mathcal{P}_L)) = \Lambda_L$$

is σ -dimensional, so (T1) in the Exchange Lemma holds. The limiting slow system on \mathcal{P}_L is governed by (5.68), and by the parametrization (5.79) of Λ_L it follows that (T2) holds. Also it is clear that (T3) holds with $\tau_0 = \tau_{10}$, where τ_{10} is defined in (5.85). Theorem 4.5 implies that there exists a neighborhood V_0 of q_0 such that

$$(6.100) \quad \mathcal{I}_\epsilon^* \cap V_0 \text{ is } C^1 \text{ } O(\epsilon)\text{-close to } W_0^u(\tilde{\Lambda}_L) \cap V_0,$$

where $\mathcal{I}_\epsilon^* = \mathcal{I}_\epsilon \cdot [0, \infty)$. Similarly,

$$(6.101) \quad \mathcal{J}_\epsilon^* \cap V_0 \text{ is } C^1 \text{ } O(\epsilon)\text{-close to } W_0^s(\tilde{\Lambda}_R) \cap V_0,$$

where $\mathcal{J}_\epsilon^* = \mathcal{J}_\epsilon \cdot (-\infty, 0]$. From Proposition 5.4, it follows that the projections of \mathcal{I}_ϵ^* and \mathcal{J}_ϵ^* in the 5-dimensional space $\{r = 0\}$ intersect transversally at a unique point in Γ near q_0 . For the relation $r = \exp(-\kappa/\epsilon)$, we then recover a unique intersection point

$$q_\epsilon \in \mathcal{I}_\epsilon^* \cap \mathcal{J}_\epsilon^* \cap \Gamma$$

in $(\beta, r, w_1, w_2, \xi, \kappa)$ -space. By construction we have (6.96) and (6.97). The estimates (6.98) follows from (4.53). Note that

$$\max_{\sigma_1 \cup \gamma_0 \cup \sigma_2} \kappa = \kappa_0,$$

where σ_1, σ_2 and γ_0 are defined in Section 5.3, so we obtain (6.99). \square

Proposition 6.2. *Assume (H1)-(H3) hold. Let $q_\epsilon = (\beta_\epsilon^0, r_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0, \xi_\epsilon^0, \kappa_\epsilon^0) \in \Gamma$ be defined in Proposition 6.1. Let $v_\epsilon^0 = \exp(\kappa_\epsilon^0/\epsilon)$ and*

$$(6.102) \quad (\tilde{\beta}_\epsilon, \tilde{v}_\epsilon, \tilde{w}_{1\epsilon}, \tilde{w}_{2\epsilon})(\xi) = (\beta_\epsilon^0, v_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0) \underset{(2.11\epsilon)}{\bullet} (\xi - \xi_\epsilon^0),$$

or equivalently,

$$(6.103) \quad (\tilde{\beta}_\epsilon, \tilde{v}_\epsilon, \tilde{w}_{1\epsilon}, \tilde{w}_{2\epsilon}, \xi) = (\beta_\epsilon^0, v_\epsilon^0, w_{1\epsilon}^0, w_{2\epsilon}^0, \xi_\epsilon^0) \underset{(2.14\epsilon)}{\bullet} \left(\frac{\xi - \xi_\epsilon^0}{\epsilon} \right).$$

Then $(\tilde{\beta}_\epsilon, \tilde{v}_\epsilon)$ is a solution of (2.9 ϵ) and (2.10), and it satisfies (2.20).

Proof. Since (2.9 ϵ) and (2.11 ϵ) are equivalent, and $(\tilde{\beta}_\epsilon, \tilde{v}_\epsilon, \tilde{w}_{1\epsilon}, \tilde{w}_{2\epsilon})(\xi)$ is a solution of (2.11 ϵ), we know $(\tilde{\beta}_\epsilon, \tilde{v}_\epsilon)$ is a solution of (2.9 ϵ).

Let $T_{1\epsilon}, T_{2\epsilon} \in \mathbb{R}$ be defined in Proposition 6.1. Then

$$q_\epsilon \underset{(5.64\epsilon)}{\bullet} (-T_{1\epsilon}/\epsilon) \in \mathcal{I}_\epsilon, \quad q_\epsilon \underset{(5.64\epsilon)}{\bullet} (T_{2\epsilon}/\epsilon) \in \mathcal{J}_\epsilon.$$

Since $\mathcal{I}_\epsilon \subset W_\epsilon^u(\mathcal{U}_L)$ and $\mathcal{J}_\epsilon \subset W_\epsilon^s(\mathcal{U}_R)$, from (5.61) and (5.63) we have

$$\lim_{t \rightarrow -\infty} \text{dist}(q_\epsilon \underset{(5.64\epsilon)}{\bullet} t, \mathcal{U}_L) = 0, \quad \lim_{t \rightarrow \infty} \text{dist}(q_\epsilon \underset{(5.64\epsilon)}{\bullet} t, \mathcal{U}_R) = 0,$$

which implies (2.10). Since $\kappa_\epsilon = \epsilon \log(v_\epsilon)$, from (6.99) we obtain (2.20). \square

6.2. Convergence of Trajectories. Based on the results in Proposition 6.1, we first derive some estimates for the self-similar solution $\tilde{u}_\epsilon(\xi)$.

Proposition 6.3. *Let $\tilde{u}_\epsilon = (\tilde{\beta}_\epsilon, \tilde{v}_\epsilon)$ be the solution of (2.9 ϵ) and (2.10) in Proposition 6.2. Let p_ϵ^{in} and p_ϵ^{out} be defined in Proposition 6.1. Then*

$$(6.104) \quad |\xi_\epsilon^{\text{in}} - s| + |\xi_\epsilon^{\text{out}} - s| = o(1)$$

$$(6.105) \quad \int_{-\infty}^{\xi_\epsilon^{\text{in}}} |\tilde{u}(\xi) - u_L| d\xi + \int_{\xi_\epsilon^{\text{out}}}^{\infty} |\tilde{u}(\xi) - u_R| d\xi = o(1)$$

$$(6.106) \quad \int_{\xi_\epsilon^{\text{in}}}^{\xi_\epsilon^{\text{out}}} \tilde{u}(\xi) d\xi = (0, e_0) + o(1)$$

as $\epsilon \rightarrow 0$, where ξ_ϵ^{in} and $\xi_\epsilon^{\text{out}}$ are ξ -coordinates of p_ϵ^{in} and p_ϵ^{out} , respectively.

Proof. Note that s is the ξ -coordinate of p_0^{in} , so

$$|\xi_\epsilon^{\text{in}} - s| \leq |p_\epsilon^{\text{in}} - p_0^{\text{in}}|,$$

which tends to zero by (6.97). Similarly, $|\xi_\epsilon^{\text{out}} - s| \rightarrow 0$. This gives (6.104).

Since every point in \mathcal{U}_L has u -coordinate equal to u_L ,

$$|\tilde{u}(\xi) - u_L| \leq \text{dist}((\tilde{u}(\xi), \tilde{w}(\xi), \xi), \mathcal{U}_L) = \text{dist}((u_\epsilon^0, w_\epsilon^0, \xi_\epsilon^0) \underset{(2.12\epsilon)}{\bullet} \frac{\xi - \xi_\epsilon^0}{\epsilon}, \mathcal{U}_L),$$

where the last equality follows from (6.103). Using (5.61), the last term is $\leq C \exp(\nu \frac{\xi - \xi_\epsilon^0}{\epsilon})$. Since $\xi_\epsilon^{\text{in}} < \xi_\epsilon^0$, it follows that

$$\int_{-\infty}^{\xi_\epsilon^{\text{in}}} |\tilde{u}(\xi) - u_L| d\xi \leq \int_{-\infty}^{\xi_\epsilon^{\text{in}}} C \exp(\nu \frac{\xi - \xi_\epsilon^0}{\epsilon}) d\xi \leq \int_{-\infty}^{\xi_\epsilon^0} C \exp(\nu \frac{\xi - \xi_\epsilon^0}{\epsilon}) d\xi = \frac{\epsilon}{\nu} C,$$

A similar inequality holds for $\int_{\xi_\epsilon^{\text{out}}}^{\infty} |\tilde{u}(\xi) - u_R| d\xi$, so we obtain (6.105).

Since $\tilde{\beta}_\epsilon(\xi)$ is uniformly bounded in ϵ , its integral between ξ_ϵ^{in} and $\xi_\epsilon^{\text{out}}$ is $o(1)$ by (6.104), and this proves the first part of (6.106). From the equation of ξ in (5.64 ϵ), denoting the time variable by ζ , we can write $\xi = \xi(\zeta)$ by

$$(6.107) \quad \xi(0) = \xi_\epsilon^0, \quad \frac{d\xi}{d\zeta} = \epsilon \tilde{r}_\epsilon(\xi),$$

where $\tilde{r}_\epsilon(\xi) = 1/\tilde{v}_\epsilon(\xi)$. From (6.96) we have

$$(6.108) \quad \xi(-T_{1\epsilon}) = \xi_\epsilon^{\text{in}}, \quad \xi(T_{2\epsilon}) = \xi_\epsilon^{\text{out}}.$$

From (6.107) and (6.108) it follows that

$$\int_{\xi_\epsilon^{\text{in}}}^{\xi_\epsilon^{\text{out}}} \tilde{v}(\xi) d\xi = \int_{\xi_\epsilon^{\text{in}}}^{\xi_\epsilon^{\text{out}}} \frac{1}{\tilde{r}(\xi)} d\xi = \int_{-T_{1\epsilon}}^{T_{2\epsilon}} \epsilon d\zeta = \epsilon(T_{1\epsilon} + T_{2\epsilon}),$$

which converges to $w_{2L} - w_{2R} = e_0$ by (6.98). This proves (6.106). \square

From the estimates in Proposition 6.3, we can derive the weak convergence of $\tilde{u}(\xi)$ as follows.

Proposition 6.4. *Let $\tilde{u}_\epsilon = (\tilde{\beta}_\epsilon, \tilde{v}_\epsilon)$ be the solution of (2.9 ϵ) and (2.10) given in Proposition 6.2. Then*

$$(6.109) \quad \tilde{u}_\epsilon \rightharpoonup u_L + (u_R - u_L)H(\xi - s) + (0, e_0)\delta_0(\xi - s)$$

in the sense of distributions as $\epsilon \rightarrow 0$.

Proof. Let $\psi \in C_c^\infty(\mathbb{R})$ be a smooth function with compact support. From (6.105) we have

$$\left| \int_{-\infty}^{\xi_\epsilon^{\text{in}}} \psi(\xi) (\tilde{u}(\xi) - u_L) d\xi \right| \leq \|\psi\|_{L^\infty} \int_{-\infty}^{\xi_\epsilon^{\text{in}}} |\tilde{u}(\xi) - u_L| d\xi \leq \|\psi\|_{L^\infty} C\epsilon,$$

which implies

$$\int_{-\infty}^{\xi_\epsilon^{\text{in}}} \psi(\xi) \tilde{u}(\xi) d\xi = \left(\int_{-\infty}^{\xi_\epsilon^{\text{in}}} \psi(\xi) d\xi \right) u_L + o(1) = \left(\int_{-\infty}^s \psi(\xi) d\xi \right) u_L + o(1).$$

A similar inequality holds for $\int_{\xi_\epsilon^{\text{out}}}^{\infty} \psi u d\xi$, so

$$(6.110) \quad \int_{\mathbb{R} \setminus [\xi_\epsilon^{\text{in}}, \xi_\epsilon^{\text{out}}]} \psi(\xi) \tilde{u}(\xi) d\xi = \left(\int_{-\infty}^s \psi(\xi) d\xi \right) u_L + \left(\int_s^\infty \psi(\xi) d\xi \right) u_R + o(1).$$

On the other hand, from (6.104) and (6.106) we have

$$\begin{aligned} \int_{\xi_\epsilon^{\text{in}}}^{\xi_\epsilon^{\text{out}}} |(\psi(\xi) - \psi(s)) \tilde{u}_\epsilon(\xi)| d\xi &\leq \left(\max_{\xi \in [\xi_\epsilon^{\text{in}}, \xi_\epsilon^{\text{out}}]} |\psi(\xi) - \psi(s)| \right) \left(\int_{\xi_\epsilon^{\text{in}}}^{\xi_\epsilon^{\text{out}}} \tilde{u}_\epsilon(\xi) d\xi \right) \\ &= o(1) \left((0, e_0) + o(1) \right) = o(1) \end{aligned}$$

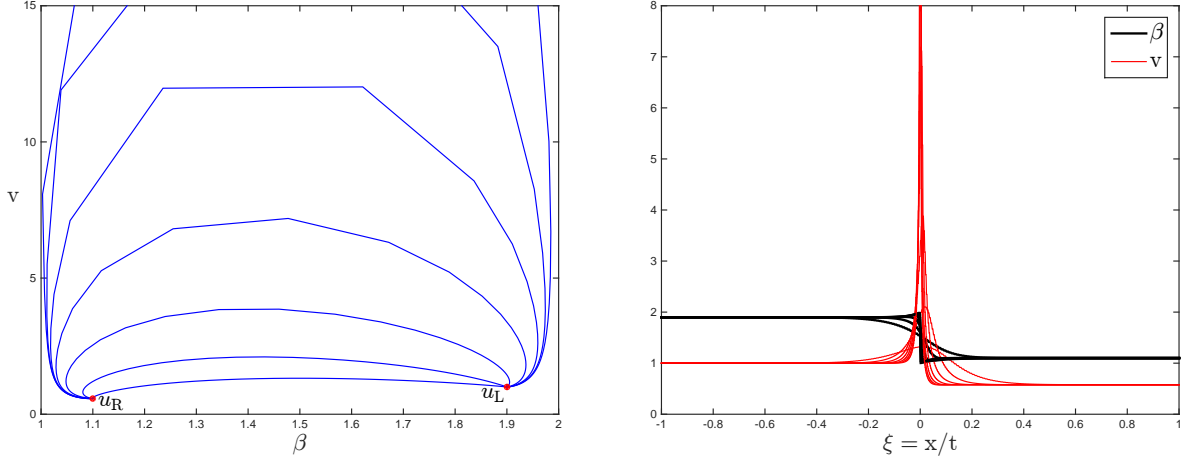


FIGURE 7. Lax-Friedrichs Scheme up to 20,000 steps with CFL=0.05

which implies

$$(6.111) \quad \int_{\xi_{\epsilon}^{\text{in}}}^{\xi_{\epsilon}^{\text{out}}} \psi(\xi) \tilde{u}_{\epsilon}(\xi) d\xi = \psi(s) \int_{\xi_{\epsilon}^{\text{in}}}^{\xi_{\epsilon}^{\text{out}}} \tilde{u}_{\epsilon}(\xi) d\xi + o(1) = \psi(s)(0, e_0) + o(1).$$

Combining (6.110) and (6.111) we obtain

$$(6.112) \quad \int_{-\infty}^{\infty} \psi(\xi) \tilde{u}_{\epsilon}(\xi) d\xi = u_L \int_{-\infty}^s \psi(\xi) d\xi + u_R \int_s^{\infty} \psi(\xi) d\xi + (0, e_0)\psi(s) + o(1).$$

This holds for all ψ , so we have (6.109). \square

Converting the results of Proposition 6.4 from self-similar variables to physical space variables, we obtain the following

Proposition 6.5. *Let $\tilde{u}_{\epsilon} = (\tilde{\beta}_{\epsilon}, \tilde{v}_{\epsilon})$ be the solution of (2.9 ϵ) and (2.10) given in Proposition 6.2. Let $u_{\epsilon}(x, t) = \tilde{u}_{\epsilon}(x/t)$. Then the weak convergence (2.21a) and (2.21b) holds.*

Proof. Let $\varphi \in C_c^{\infty}(\mathbb{R} \times \mathbb{R}_+)$. From (6.109) we have

$$\begin{aligned} \int_0^{\infty} \int_{-\infty}^{\infty} \varphi(x, t) u_{\epsilon}(x, t) dx dt &= \int_0^{\infty} t \int_{-\infty}^{\infty} \varphi(t\xi, t) \tilde{u}_{\epsilon}(\xi) d\xi dt \\ &= \int_0^{\infty} t \left\{ u_L \int_{-\infty}^s \varphi(t\xi, t) d\xi + (0, e_0) \varphi(st, t) + u_R \int_s^{\infty} \varphi(t\xi, t) d\xi \right\} dt + o(1) \\ &= u_L \int_0^{\infty} \int_{-\infty}^{st} \varphi(x, t) dx dt + u_R \int_0^{\infty} \int_{st}^{\infty} \varphi(x, t) dx dt + (0, e_0) \int_0^{\infty} t \varphi(st, t) dt + o(1). \end{aligned}$$

From (2.22), this means (2.21) holds. \square

Now Propositions 6.2 and 6.5 complete the proof of the Main Theorem.

7. NUMERICAL SIMULATIONS

Some numerical solutions for (1.1) using the Lax-Friedrichs scheme are shown in Figure 7. The solutions appear to grow unboundedly as the number of steps increases.

Also some numerical approximations for (2.8 ϵ) are shown in Fig 8. The algorithm was a shooting method following the descriptions in [KSS03]. Since w_1 and ξ are essentially constant near the shock, we project the trajectories in the (β, r, w_2) space. Note that $w_2(\xi)$ does not converge as $\xi \rightarrow \pm\infty$ while $x_2 = w_2 + (\xi - s)v$ converges, we replace w_2 by x_2 (again, following [KSS03]). Note that x_2 is a mild modification of w_2 near the shock since within the ϵ -neighborhood of $\xi = s$ the difference between x_2 and w_2 is of order $o(1)$.

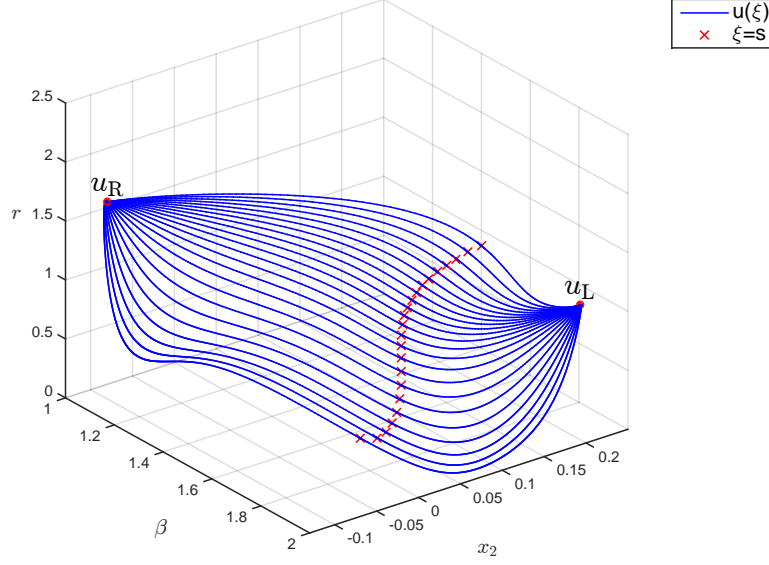


FIGURE 8. Trajectories for $\epsilon u'' = (Df(u) - s)u'$ as ϵ decreases from 1 to 0.01. The variable x_2 is a modification of w_2 .

As ϵ decreases, the minimal value of r -coordinate on the trajectories in Fig 8 tends to zero. This means the maximum of v tends to infinity. Also observe that the change of the value of x_2 concentrates in the vicinity of $\beta = \rho_1$ and $\beta = \rho_2$. This is consistent with our proof for the main theorem.

8. PROOF OF PROPOSITION 3.3

If the section we prove the sufficiency of the conditions in Proposition 3.3 for (H3), which says that there is a trajectory for (2.15) connecting (u_L, w_L, s) and $\{\beta = \rho_1, v = +\infty\}$, and a trajectory connecting (u_R, w_R, s) and $\{\beta = \rho_2, v = +\infty\}$. We will focus on finding the first trajectory while finding the second one is similar.

We will switch back and forth between (β, v) - and (β, r) -coordinates, where $r = 1/v$. The system (2.15) is converted to (5.65) in (β, r) -coordinates. It suffices to find trajectories connecting $u_L = (\beta_L, r_L)$ and $p_L \equiv (\rho_1, 0)$ for (5.65) with $(w, \xi) = (w_L, s)$. From (H1) we know that u_L is a source, and we will also see that p_L is a saddle. Our strategy is to construct a negatively invariant region in which every trajectory goes backward to u_L , and one of those trajectories goes forward to p_L . See Fig 9.

To construct a such region, we first study the flow on the boundary of the feasible region $\{\rho_2 \leq \beta \leq \rho_1\}$. The equation of $\dot{\beta}$ in (2.15) with $(w, \xi) = (w_L, s)$ is

$$\dot{\beta} = -s\rho_1 - w_{1L} \quad \text{on } \{\beta = \rho_1\}.$$

Hence the proposition below implies that the region $\{\beta \leq \rho_1\}$ is negatively invariant. Similarly, for (2.15) with $(w, \xi) = (w_R, s)$, the region $\{\beta \geq \rho_2\}$ is positively invariant.

Lemma 8.1. *If (H1) holds, then*

$$(8.113) \quad s\rho_1 + w_{1L} < 0 \quad \text{and} \quad s\rho_2 + w_{1R} < 0,$$

where s , w_{1L} and w_{1R} are as defined in (2.17) and (2.18).

Proof. By definition of s and w_{1L} we have

$$\begin{aligned} s\rho_1 + w_{1L} &= s\rho_1 + v_L B_1(\beta_L) - s\beta_L \\ &= \frac{v_L B_1(\beta_L) - v_R B_1(\beta_R)}{\beta_L - \beta_R} (\rho_1 - \beta_L) + v_L B_1(\beta_L). \end{aligned}$$

From Proposition 3.1, we know (H1) implies $\beta_R < \beta_L$ and

$$v_R \leq v_L \left(\frac{B_1(\beta_L) - 2B_2(\beta_L)(\beta_L - \beta_R)}{B_1(\beta_R)} \right).$$

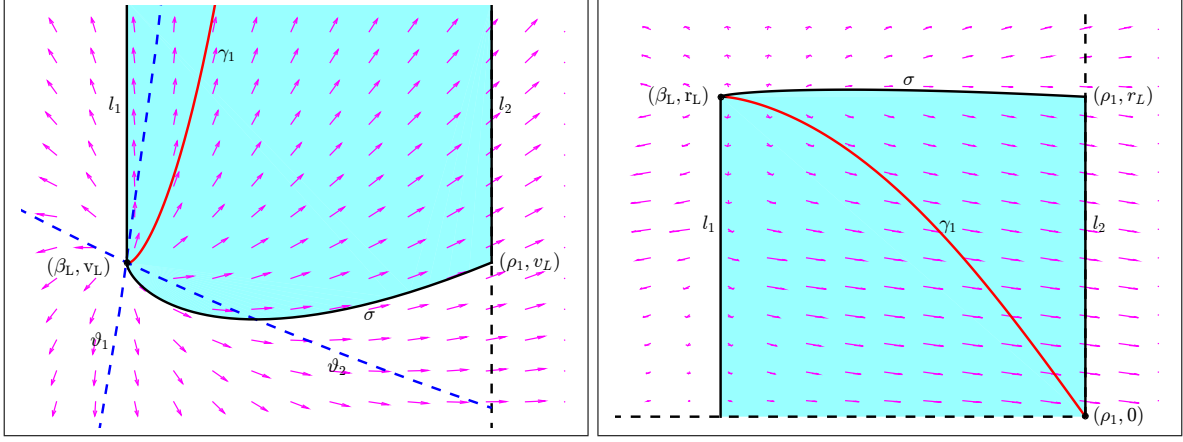


FIGURE 9. Phase portraits for (8.114) in (β, v) space and (8.117) in (β, r) space. The shaded region V is a backward invariant region in which every backward trajectory tends to u_L , and γ_1 is the unique trajectory in V which tends to $p_L = (\rho_1, 0)$.

Since $B_1(\beta_L) < 0$, it follows that

$$\begin{aligned} s\rho_1 + w_{1L} &\leq \frac{\rho_1 - \beta_L}{\beta_L - \beta_R} v_L (2B_2(\beta_L)(\beta_L - \beta_R)) + v_L B_1(\beta_L) \\ &= \left(\frac{(\rho_1 - \beta_L)(\beta_L^2 - \rho_1\rho_2)}{\beta_L^2} + \frac{(\rho_1 - \beta_L)(\rho_2 - \beta_L)}{\beta_L} \right) v_L \\ &= \frac{-\rho_2(\rho_1 - \beta_L)^2 v_L}{\beta_L^2} < 0. \end{aligned}$$

Similarly, using $s\rho_2 + w_{1R} = s\rho_2 + v_R B_1(\beta_R) - s\beta_R$, one obtains $s\rho_2 + w_{1R} < 0$. \square

Proposition 3.3. Suppose (H1) holds. If $\beta_R < \sqrt{\rho_1\rho_2} < \beta_L$, $w_{10} < 0$, $w_{2R} < 0 < w_{2L}$, and $|s|$ is sufficiently small, then (H3) holds.

Proof. We focus on u_L while the proof for u_R is similar. As mentioned at the beginning of this section and Fig 9, we first construct a negatively invariant region in which every trajectory goes backward to u_L , and then show that one of those trajectories goes forward to p_L .

Consider (2.15) with $(w, \xi) = (w_L, s)$. That is,

$$\begin{aligned} (8.114) \quad \dot{\beta} &= vB_1(\beta) - s\beta - w_{1L} \\ \dot{v} &= v^2 B_2(\beta) - sv - w_{2L}. \end{aligned}$$

The null-clines for this system are

$$(8.115) \quad \dot{\beta} = 0 : \quad v = \vartheta_1(\beta) := \frac{\beta(s\beta + w_{1L})}{(\beta - \rho_1)(\beta - \rho_2)}$$

$$(8.116) \quad \dot{v} = 0 : \quad v = \vartheta_2(\beta) := \frac{s\beta + \sqrt{s^2\beta^2 + 2w_{2L}(\beta^2 - \rho_1\rho_2)}}{\beta^2 - \rho_1\rho_2} \beta.$$

When $|s|$ is small, it can be readily seen that ϑ_1 is increasing and ϑ_2 is decreasing on the interval $(\sqrt{\rho_1\rho_2}, \rho_1)$. Also we have $\vartheta_2(\rho_1) > 0$ since $w_{2L} > 0$.

Let $\sigma(\tau)$ be the solution to (8.114) with initial condition $\sigma(0) = (\rho_1, v_L)$. By the monotonicity of ϑ_1 and ϑ_2 , we know that σ hits the half-line $l_1 = \{(\beta_L, v) : v \geq v_L\}$ at some time $\tau_- < 0$. Let $l_2 = \{(\rho_1, v) : v \geq v_L\}$ and V be the region enclosed by the curves

$$l_1 \cup \{\sigma(\tau) : \tau_- \leq \tau \leq 0\} \cup l_2.$$

Then V forms a backward invariant region. See Fig 9.

We claim that u_L attracts every point in V in backward time. Note that

$$\begin{aligned} & \frac{\partial}{\partial \beta}(vB_1(\beta) - s\beta - w_{10}) + \frac{\partial}{\partial v}(v^2B_2(\beta) - sv - w_{2L}) \\ &= vB_1'(\beta) - s + 2vB_2(\beta) - s \\ &= 4vB_2(\beta) - 2s, \end{aligned}$$

which is positive when $\beta \in (\sqrt{\rho_1\rho_2}, \rho_1)$ and s is small. In the last equality we used $B_1'(\beta) = 2B_2(\beta)$. By Bendixson's negative criterion, the system has no periodic orbit inside V . Since V is backward invariant and u_L is the only equilibrium on the closure of V , it follows from the Poincaré-Bendixson Theorem that every trajectory in V tends to u_L in backward time.

It remains to show that there is a trajectory in V tending to $\{\beta = \rho_1, v = \infty\}$. Let $r = 1/v$. Then (8.114) is converted to, after multiplying by r ,

$$(8.117) \quad \begin{aligned} \dot{\beta} &= B_1(\beta) - s\beta r - w_{10}r \\ \dot{r} &= -rB_2(\beta) + sr^2 + w_{2L}r^3. \end{aligned}$$

At the equilibrium $p_L = (\rho_1, 0)$, the eigenvalues of the linearized system are $\lambda_+ = 1 - \frac{\rho_2}{\rho_1}$ and $\lambda_- = \frac{-1}{2}(1 - \frac{\rho_2}{\rho_1})$, and the corresponding eigenvectors are $y_+ = (1, 0)$ and $y_- = (\frac{2\rho_1(w_{1L} + s\rho_1)}{3(\rho_1 + \rho_2)}, 1)$, so p_L is a hyperbolic saddle, and hence there exists a trajectory, denoted by γ_1 , which tends to p_L . The trajectory of γ_1 is tangent to the line $\{p_L + ty_- : t \in \mathbb{R}\}$ at p_L . Since $s\rho_1 + w_{1L} < 0$ by Lemma 8.1, we know $p_L + ty_1$, $t \geq 0$, lies in the region $\{\beta < \rho_1, r \geq 0\}$. Therefore, converting (8.117) back to (8.114), the solution converted from $\gamma_1(\tau)$, also denoted by $\gamma_1(\tau)$, lies in V . Now we conclude that $\gamma_1(\tau)$ approaches $\{\beta = \rho_1, v = \infty\}$ in forward time and approaches u_L in backward time. \square

ACKNOWLEDGMENTS

The author wishes to thank Professor Maciej Krupa for his fruitful comments. Also the author wishes to express his gratitude to his advisor, Professor Barbara L. Keyfitz, for her constant patience and encouragement, and for the inspiration he received from their many stimulating conversations, all of which contributed to the completion of this work.

REFERENCES

- [Bru99] Pavol Brunovský. C^r -inclination theorems for singularly perturbed equations. *J. Differential Equations*, 155(1):133–152, 1999.
- [CL88] Shui-Nee Chow and Kening Lu. C^k centre unstable manifolds. *Proc. Roy. Soc. Edinburgh Sect. A*, 108(3-4):303–320, 1988.
- [CL03] Gui-Qiang Chen and Hailiang Liu. Formation of δ -shocks and vacuum states in the vanishing pressure limit of solutions to the Euler equations for isentropic fluids. *SIAM J. Math. Anal.*, 34(4):925–938, 2003.
- [CLY00a] Shui-Nee Chow, Weishi Liu, and Yingfei Yi. Center manifolds for invariant sets. *J. Differential Equations*, 168(2):355–385, 2000. Special issue in celebration of Jack K. Hale's 70th birthday, Part 2 (Atlanta, GA/Lisbon, 1998).
- [CLY00b] Shui-Nee Chow, Weishi Liu, and Yingfei Yi. Center manifolds for smooth invariant manifolds. *Trans. Amer. Math. Soc.*, 352(11):5179–5211, 2000.
- [Daf10] Constantine M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010.
- [Den90] Bo Deng. Homoclinic bifurcations with nonhyperbolic equilibria. *SIAM J. Math. Anal.*, 21(3):693–720, 1990.
- [DP99] Donald A. Drew and Stephen L. Passman. *Theory of multicomponent fluids*, volume 135 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1999.
- [DR96] Freddy Dumortier and Robert Roussarie. Canard cycles and center manifolds. *Mem. Amer. Math. Soc.*, 121(577):x+100, 1996. With an appendix by Cheng Zhi Li.
- [Fen77] Neil Fenichel. Asymptotic stability with rate conditions. II. *Indiana Univ. Math. J.*, 26(1):81–93, 1977.
- [Fen79] Neil Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Differential Equations*, 31(1):53–98, 1979.
- [Fen74] Neil Fenichel. Asymptotic stability with rate conditions. *Indiana Univ. Math. J.*, 23:1109–1137, 1973/74.
- [Har64] Philip Hartman. *Ordinary differential equations*. John Wiley & Sons, Inc., New York-London-Sydney, 1964.
- [HPS77] M. W. Hirsch, C. C. Pugh, and M. Shub. *Invariant manifolds*. Lecture Notes in Mathematics, Vol. 583. Springer-Verlag, Berlin-New York, 1977.
- [JK94] C. K. R. T. Jones and N. Kopell. Tracking invariant manifolds with differential forms in singularly perturbed systems. *J. Differential Equations*, 108(1):64–88, 1994.

- [JKK96] C. K. R. T. Jones, Tasso J. Kaper, and Nancy Kopell. Tracking invariant manifolds up to exponentially small errors. *SIAM J. Math. Anal.*, 27(2):558–577, 1996.
- [JKL91] C. Jones, N. Kopell, and R. Langer. Construction of the FitzHugh-Nagumo pulse using differential forms. In *Patterns and dynamics in reactive media (Minneapolis, MN, 1989)*, volume 37 of *IMA Vol. Math. Appl.*, pages 101–115. Springer, New York, 1991.
- [Jon95] Christopher K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical systems (Montecatini Terme, 1994)*, volume 1609 of *Lecture Notes in Math.*, pages 44–118. Springer, Berlin, 1995.
- [JT09] Christopher K. R. T. Jones and Siu-Kei Tin. Generalized exchange lemmas and orbits heteroclinic to invariant manifolds. *Discrete Contin. Dyn. Syst. Ser. S*, 2(4):967–1023, 2009.
- [Kap99] Tasso J. Kaper. An introduction to geometric methods and dynamical systems theory for singular perturbation problems. In *Analyzing multiscale phenomena using singular perturbation methods (Baltimore, MD, 1998)*, volume 56 of *Proc. Sympos. Appl. Math.*, pages 85–131. Amer. Math. Soc., Providence, RI, 1999.
- [Key11] Barbara Lee Keyfitz. Singular shocks: retrospective and prospective. *Confluentes Math.*, 3(3):445–470, 2011.
- [KJ01] Tasso J. Kaper and Christopher K. R. T. Jones. A primer on the exchange lemma for fast-slow systems. In *Multiple-time-scale dynamical systems (Minneapolis, MN, 1997)*, volume 122 of *IMA Vol. Math. Appl.*, pages 65–87. Springer, New York, 2001.
- [KS01a] M. Krupa and P. Szmolyan. Extending geometric singular perturbation theory to nonhyperbolic points—fold and canard points in two dimensions. *SIAM J. Math. Anal.*, 33(2):286–314 (electronic), 2001.
- [KS01b] M. Krupa and P. Szmolyan. Relaxation oscillation and canard explosion. *J. Differential Equations*, 174(2):312–368, 2001.
- [KSS03] Barbara Lee Keyfitz, Richard Sanders, and Michael Sever. Lack of hyperbolicity in the two-fluid model for two-phase incompressible flow. *Discrete Contin. Dyn. Syst. Ser. B*, 3(4):541–563, 2003. Nonlinear differential equations, mechanics and bifurcation (Durham, NC, 2002).
- [KSZ04] Barbara Lee Keyfitz, Michael Sever, and Fu Zhang. Viscous singular shock structure for a nonhyperbolic two-fluid model. *Nonlinearity*, 17(5):1731–1747, 2004.
- [KT12] Barbara Lee Keyfitz and Charis Tsikkou. Conserving the wrong variables in gas dynamics: a Riemann solution with singular shocks. *Quart. Appl. Math.*, 70(3):407–436, 2012.
- [Liu00] Weishi Liu. Exchange lemmas for singular perturbation problems with certain turning points. *J. Differential Equations*, 167(1):134–180, 2000.
- [Liu04] Weishi Liu. Multiple viscous wave fan profiles for Riemann solutions of hyperbolic systems of conservation laws. *Discrete Contin. Dyn. Syst.*, 10(4):871–884, 2004.
- [LVV06] Weishi Liu and Erik Van Vleck. Turning points and traveling waves in FitzHugh-Nagumo type equations. *J. Differential Equations*, 225(2):381–410, 2006.
- [RT02] Jonathan E. Rubin and David Terman. Geometric singular perturbation analysis of neuronal dynamics. In *Handbook of dynamical systems, Vol. 2*, pages 93–146. North-Holland, Amsterdam, 2002.
- [Sch04] Stephen Schecter. Existence of Dafermos profiles for singular shocks. *J. Differential Equations*, 205(1):185–210, 2004.
- [Sch08a] Stephen Schecter. Exchange lemmas. I. Deng’s lemma. *J. Differential Equations*, 245(2):392–410, 2008.
- [Sch08b] Stephen Schecter. Exchange lemmas. II. General exchange lemma. *J. Differential Equations*, 245(2):411–441, 2008.
- [Sev07] Michael Sever. Distribution solutions of nonlinear systems of conservation laws. *Mem. Amer. Math. Soc.*, 190(889):viii+163, 2007.
- [Sil67] L. P. Silnikov. Existence of a countable set of periodic motions in a four-dimensional space in an extended neighborhood of a saddle-focus. *Dokl. Akad. Nauk SSSR*, 172:54–57, 1967.
- [Smo83] Joel Smoller. *Shock waves and reaction-diffusion equations*, volume 258 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science]*. Springer-Verlag, New York-Berlin, 1983.
- [Tin94] Siu-Kei Tin. *On the dynamics of tangent spaces near a normally hyperbolic invariant manifold*. ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Ph.D.)—Brown University.
- [TZZ94] De Chun Tan, Tong Zhang, and Yu Xi Zheng. Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws. *J. Differential Equations*, 112(1):1–32, 1994.
- [VvG87] A. Vanderbauwhede and S. A. van Gils. Center manifolds and contractions on a scale of Banach spaces. *J. Funct. Anal.*, 72(2):209–224, 1987.
- [Wig94] Stephen Wiggins. *Normally hyperbolic invariant manifolds in dynamical systems*, volume 105 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994. With the assistance of György Haller and Igor Mezić.

DEPARTMENT OF MATHEMATICS, THE OHIO STATE UNIVERSITY, COLUMBUS, OH 43210
 E-mail address: hsu.296@osu.edu